Fourier Series in L^2

A Brief Introduction

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Abstract

Based on Fourier's idea of writing functions as trigonometric series, we first find the explicit formula of the Fourier Coefficients by assuming that the convergence exists. We understand these coefficients in the context of an inner product space and give two examples. Moreover, we investigate the pointwise convergence of Fourier series of continuous functions. Then, we pay closer attention to the more general result of the convergence of Fourier series in L^2. As our final takeaway, we conclude that the convergence of Fourier series is less robust than that of power series, and appreciate the nice properties of the Hilbert space L^2.

Contents

1 Introduction

Since the arrival of power series, physicists have found great advantage of understanding complicated functions as infinite series of familiar functions with tuned coefficients. However, the convergence of power series is generally hard to satisfy and the harmonic motions in mechanics and thermodynamics inherently request the usage of trigonometric series.

In Joseph Fourier's 1822 treatise [3], "The Analytical Theory of Heat", he boldly claims that "There is no function $f(x)$ or part of a function, which cannot be expressed by a trigonometric series." While physicists were at ease using Fourier's trigonometric series to analyze their functions, mathematicians at the nineteenth century were challenged by Fourier's bold assertion and determined to attach mathematical rigor to the types of functions that can be "expressed" by trigonometric series. Indeed, the creation of Lebesgue measure and the Lebesgue integral was motivated by Fourier's assertion.

Similar to the definition of power series, trigonometric series are defined as an infinite sum of trigonometric functions as components.

Definition 1.1. *A trigonometric series is defined as*

$$a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx).$$

In this series, the functions,

$$\{1, \cos(x), \sin(x), \cos(2x), \sin(2x), \cos(3x), \sin(3x), \ldots\}$$

serve as the components.

Therefore, to investigate the question asked by Fourier, we will first assert that there is a sequence of real numbers $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$, such that

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx), \tag{1}$$

where the infinite series is assumed to be convergent. We make this assumption to find a candidate for these two sequences. After we have found these coefficients, we note that "convergence" can vary in different contexts. In our paper, we will discuss pointwise convergence and convergence in L^2 space.

2 Trigonometric Series

2.1 Periodicity

One nice property of the class of trigonometric functions like cos and sin is their common period of 2π. Therefore, for $\cos(nx)$, and $\sin(mx)$ with arbitrary $m, n \in \mathbb{N}$, they must be periodic on the interval $(-\pi, \pi]$.

Then, in the following discussions on orthogonality, Fourier coefficients, and convergence tests, we restrict our attention to the behavior of the series on the interval $(-\pi, \pi]$

$$f(x) = f(x + 2k\pi),\ k \in \mathbb{Z},$$

by the periodicity of f.

Note that the interpretation of functions as abstract mappings between inputs and outputs was not the common in Fourier's time. Conventionally, functions were interpreted as an universal formula that governs the behavior of functions on the entire domain. In this context, Fourier's treatment of trigonometry series in one period had some controversies. To avoid confusions, Fourier gave his understanding of functions as "a succession of values or ordinates, each of which is arbitrary" [3]. With such a general definition of functions, the functional values do not subject to a common formula, which validates studying series in only one period.

2.2 Orthogonality of Inner products of trigonometric components

We first observe that the integral of trigonometric functions have special values, which will help calculate the Fourier coefficients in the next section.

Lemma 2.1. *For all $n \in \mathbb{N}$,*

$$\int_{-\pi}^{\pi} \cos(nx)dx = 0 \quad \textit{and} \quad \int_{-\pi}^{\pi} \sin(nx)dx = 0.$$

Proof. Using Riemann integration we have that,

$$\int_{-\pi}^{\pi} \cos(nx)dx = \frac{1}{n}(\sin(n\pi) - \sin(-n\pi)) = 0$$

since $\sin(n\pi) = 0$ for all $n \in \mathbb{N}$. On the other hand,

$$\int_{-\pi}^{\pi} \sin(nx)dx = \frac{1}{n}(-\cos(n\pi) + \cos(-n\pi)) = \frac{1}{n}(-\cos(n\pi) + \cos(n\pi)) = 0.$$

since cosine is an even function. ∎

Lemma 2.2. *For all $n \in \mathbb{N}$,*

$$\int_{-\pi}^{\pi} \cos^2(nx)dx = \pi \quad \textit{and} \quad \int_{-\pi}^{\pi} \sin^2(nx)dx = \pi. \tag{2}$$

Proof. Since $\sin^2(nx)$ is integrable, we can use integration by parts with $u = \sin(nx)$ and $dv = \sin(nx)$ to get,

$$\int \sin^2(nx)dx = -\frac{1}{n}\sin(nx)\cos(nx) + \int \cos^2(nx)dx + \mathcal{C}.$$

Since $\sin^2(nx) = 1 - \cos^2(nx)$ and since $\cos^2(nx)$ is integrable,

$$\int 1 - \cos^2(nx) = -\frac{1}{n}\sin(nx)\cos(nx) + \int \cos^2(nx)dx + \mathcal{C}$$

and,

$$x - \int \cos^2(nx) = -\frac{1}{n}\sin(nx)\cos(nx) + \int \cos^2(nx)dx + \mathcal{C}$$

which means that, by solving for $\int \cos^2(nx)dx$,

$$\int \cos^2(nx) = \frac{x}{2} + \frac{1}{4n}\sin(2nx) + \mathcal{C}.$$

By replacing in the first equation,

$$\int \sin^2(nx)dx = \frac{x}{2} - \frac{1}{4n}\sin(2nx) + \mathcal{C}.$$

Hence,

$$\int_{-\pi}^{\pi} \sin^2(nx)dx = \frac{\pi}{2} - \frac{1}{4n}\sin(2n\pi) - \left(\frac{-\pi}{2} - \frac{1}{4n}\sin(-2n\pi)\right) = \pi.$$

because $\sin(2\pi n) = 0$ for all $n \in \mathbb{Z}$.

Finally,

$$\int \cos^2(nx) = \frac{\pi}{2} - \frac{1}{4n}\cos(2n\pi) - \left(\frac{-\pi}{2} - \frac{1}{4n}\cos(-2n\pi)\right) = \pi$$

because,

$$\frac{1}{4n}\cos(-2n\pi) = \frac{1}{4n}\cos(2n\pi)$$

as cosine is an even function. ∎

Notice that Lemma 2.2 can be understood more generally in the context of the inner product of the L^2 space. Recall that $\langle f, g\rangle = \int_a^b fg$, where $f, g \in L^2[a,b]$, is an inner product that satisfies the four conditions for being an inner product. Then, Lemma 2.2 is just an example of the third condition $\langle v, v\rangle \geq 0$, since the square of $\sin(nx)$ and $\cos(nx)$ are both positive.

Lemma 2.3 (Orthogonality relationship part i)**.** *For all $m, n \in \mathbb{N}$,*

$$\int_{-\pi}^{\pi} \cos(mx)\sin(nx)dx = 0. \tag{3}$$

Proof. Suppose that $m \neq n$. Let $u = \sin(nx)$ and $dv = \cos(mx)$ so that, by integration by parts,

$$\int \cos(mx)\sin(nx)dx = \frac{1}{m}\sin(mx)\sin(nx) - \frac{n}{m}\int \cos(nx)\sin(mx)dx + \mathcal{C}.$$

Let $u = \cos(nx)$ and $dx = \sin(mx)$ so that by integration by parts,

$$\int \cos(nx)\sin(mx)dx = -\frac{1}{m}\cos(nx)\cos(mx) - \frac{n}{m}\int \cos(mx)\sin(nx) + \mathcal{C}$$

and replacing this quantity in the equation above,

$$\int \cos(mx)\sin(nx)dx = \frac{1}{m}\sin(mx)\sin(nx) + \frac{n}{m^2}\cos(nx)\cos(mx) + \frac{n^2}{m^2}\int \cos(mx)\sin(nx) + \mathcal{C}.$$

Solving by the integral,

$$\int \cos(mx)\sin(nx)dx = \frac{m\sin(mx)\sin(nx) + n\cos(nx)\cos(mx)}{m^2 - n^2} + \mathcal{C}.$$

Then,

$$\begin{aligned}\int_{-\pi}^{\pi} \cos(mx)\sin(nx)dx &= \frac{m\sin(m\pi)\sin(n\pi) + n\cos(n\pi)\cos(m\pi)}{m^2 - n^2} - \frac{m\sin(-\pi m)\sin(-\pi n) + n\cos(-\pi n)\cos(-\pi m)}{m^2 - n^2} \\ &= 0.\end{aligned}$$

because $\sin(\pi n) = \sin(-\pi n) = 0$ and $\cos(n\pi) = \cos(-n\pi)$.

Now suppose that $n = m$. Then,

$$\int_{-\pi}^{\pi} \cos(nx)\sin(nx)dx = -\frac{1}{2n}(\cos^2(n\pi) - \cos^2(-n\pi)) = 0$$

because cosine is an even function. ∎

Lemma 2.4 (Orthogonality relationship part ii)**.** *For every $m, n \in \mathbb{N}$ such that $m \neq n$,*

$$\int_{-\pi}^{\pi} \cos(mx)\cos(nx)dx = 0 \quad \textit{and} \quad \int_{-\pi}^{\pi} \sin(mx)\sin(nx)dx = 0. \tag{4}$$

We omit some of the details of this proof, as they are similar to the proof of the previous lemma.

Proof. By doing integration with parts twice,

$$\int \cos(mx)\cos(nx) = \frac{1}{n}\sin(nx)\cos(mx) + \frac{m}{n}\left(-\frac{1}{n}\cos(nx)\sin(mx) - \frac{m}{n}\int \cos(nx)\cos(mx)\right) + \mathcal{C}.$$

Solving by $\int \cos(nx)\cos(mx)$ we get,

$$\int \cos(nx)\cos(mx) = \frac{n\sin(nx)\cos(mx) - m\sin(mx)\cos(nx)}{n^2 - m^2} + \mathcal{C}.$$

Thus,

$$\int_{-\pi}^{\pi} \cos(nx)\cos(mx) = \frac{n\sin(n\pi)\cos(m\pi) - m\sin(m\pi)\cos(n\pi)}{n^2 - m^2} - \frac{n\sin(-n\pi)\cos(-m\pi) - m\sin(-m\pi)\cos(-n\pi)}{n^2 - m^2}$$

Since $\sin(n\pi) = 0$ for all $n \in \mathbb{Z}$,

$$\int_{-\pi}^{\pi} \cos(nx)\cos(mx)dx = 0.$$

Similarly by integrating twice,

$$\int \sin(mx)\sin(nx) = \frac{m\sin(nx)\cos(mx) - n\sin(mx)\cos(nx)}{n^2 - m^2} + \mathcal{C}$$

so that,

$$\int_{-\pi}^{\pi} \sin(mx)\sin(nx) = \frac{m\sin(n\pi)\cos(m\pi) - n\sin(m\pi)\cos(n\pi)}{n^2 - m^2} - \frac{m\sin(-\pi n)\cos(-\pi m) - n\sin(-\pi m)\cos(-\pi n)}{n^2 - m^2}.$$

Since $\sin(n\pi) = 0$ for all $n \in \mathbb{Z}$,

$$\int_{-\pi}^{\pi} \sin(nx)\sin(mx)dx = 0.$$

■

Notice that Lemma 2.3 and 2.4 together are the orthogonality relationships for the functions $\{1, \cos(x), \sin(x), \ldots\}$ in L^2 with the inner product $\langle f, g\rangle = \int_a^b fg$. Since the inner product (dot product) of two distinct vectors in $\mathbb{R}^n$ is zero when they are orthogonal, if the inner product (integral) of two distinct functions in L^2 is zero, we call those two functions orthogonal.

Moreover, we learned in linear algebra that all vectors that are pairwise orthogonal form an orthogonal basis in a finite dimensional vector space. However, L^2 is an infinite dimensional vector space. Thus, with Lemma 2.3 and 2.4, proving the components $\cos(nx)$ and $\sin(nx)$ form an orthogonal basis is equivalent to showing that any function in L^2 has a Fourier series representation. Fortunately, as we will show later, $\{\cos(nx), \sin(nx)\}$ is indeed is an orthogonal basis for L^2.

Finally, using $\{\cos(nx), \sin(nx)\}$ as an orthogonal basis for L^2, Lemma 2.3 and 2.4 are generalized in the following notation of the inner product of L^2.

i. for all $m, n \in \mathbb{N}$, $\langle\cos(mx), \sin(nx)\rangle = 0$;

ii. for $m \neq n \in \mathbb{N}$, $\langle\cos(mx), \cos(nx)\rangle = \langle\sin(mx), \sin(nx)\rangle = 0$.

This notation will be helpful to interpret the calculation of Fourier coefficients.

2.3 Finding Fourier coefficients a_i and b_i

Lemma 2.5. *Suppose that there exists an integrable (the particular definition depends on the context) function on $[-\pi, \pi]$ such that, there exists sequences $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ with*

$$\int_{-\pi}^{\pi} f(x)dx = a_0 + \sum_{n=1}^{\infty} a_n\cos(nx) + b_n\sin(nx).$$

Then,

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx \tag{5}$$

and,

$$a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(mx)dx \quad \textit{and} \quad b_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(mx)dx.$$

Proof. By assuming that the summation can be interchanged with the integral (a practical way to find the Fourier coefficients),

$$\int_{-\pi}^{\pi} f(x) = \int_{-\pi}^{\pi}\left[a_0 + \sum_{n=1}^{\infty} a_n\cos(nx) + b_n\sin(nx)\right]dx = \pi a_0 + \sum_{n=1}^{\infty}\left(a_n\int_{-\pi}^{\pi}\cos(nx)dx + b_n\int_{-\pi}^{\pi}\sin(nx)dx\right)$$

$$= \pi a_0 + \sum_{n=1}^{\infty} a_n\cdot 0 + b_n\cdot 0 = \pi a_0,$$

where $\int_{-\pi}^{\pi}\cos(nx)dx = \int_{-\pi}^{\pi}\sin(nx)dx = 0$ by Lemma 2.1. By dividing both sides by π, we get Eq. 5.

Then, to get a_m, we evaluate the integral of $f(x)\cos(mx)$,

$$\int_{-\pi}^{\pi} f(x)\cos(mx) = \int_{-\pi}^{\pi} a_0\cos(mx)dx + \sum_{n=1}^{\infty}\left[a_n\int_{-\pi}^{\pi}\cos(mx)\cos(nx)dx + b_n\int_{-\pi}^{\pi}\cos(mx)\sin(nx)dx\right]$$

$$= 0 + a_m\int_{-\pi}^{\pi}\cos(mx)\cos(mx)dx + \sum_{n=1,n\neq m}^{\infty}\left[a_n\int_{-\pi}^{\pi}\cos(mx)\cos(nx)dx + b_n\int_{-\pi}^{\pi}\cos(mx)\sin(nx)dx\right] = a_m\pi,$$

where $\int_{-\pi}^{\pi}\cos(mx)\cos(mx)dx = \pi$ by Lemma 2, and $\int_{-\pi}^{\pi}\cos(mx)\cos(nx)dx + \int_{-\pi}^{\pi}\cos(mx)\sin(nx)dx = 0$ $(n \neq m)$ by Lemma 3 and 4, which is also the orthogonal relationship of Fourier components with respect to inner products of L^2.

Similarly, to get b_m, we evaluate the integral of $f(x)\sin(mx)$,

$$\int_{-\pi}^{\pi} f(x)\sin(mx) = \int_{-\pi}^{\pi} a_0\sin(mx)dx + \sum_{n=1}^{\infty}\left[a_n\int_{-\pi}^{\pi}\sin(mx)\cos(nx)dx + b_n\int_{-\pi}^{\pi}\sin(mx)\sin(nx)dx\right]$$

$$= 0 + b_m\int_{-\pi}^{\pi}\sin(mx)\sin(mx)dx + \sum_{n=1,n\neq m}^{\infty}\left[a_n\int_{-\pi}^{\pi}\sin(mx)\cos(nx)dx + b_n\int_{-\pi}^{\pi}\sin(mx)\sin(nx)dx\right] = b_m\pi,$$

where $\int_{-\pi}^{\pi}\sin(mx)\sin(mx)dx = \pi$ by Lemma 2, and $\int_{-\pi}^{\pi}\cos(mx)\cos(nx)dx + \int_{-\pi}^{\pi}\cos(mx)\sin(nx)dx = 0$ $(n \neq m)$ by the orthogonal relationship. ■

In the context of L^2 spaces with the inner product, Fourier coefficients can be expressed in the following ways:

i. $a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx = \dfrac{\langle f, 1\rangle}{\langle 1, 1\rangle}$;

ii. $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)dx = \dfrac{\langle f, \cos(nx)\rangle}{\langle \cos(nx), \cos(nx)\rangle}$;

iii. $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)dx = \dfrac{\langle f, \sin(nx)\rangle}{\langle \sin(nx), \sin(nx)\rangle}$.

Here, the Fourier coefficients of a certain function can be understood algebraically as the projection of the function onto the presumed basis $\{1, \cos(x), \sin(x), \ldots\}$ with respect to the inner product. Thus, the a_i and b_i are the coordinates of the function with respect to $\{1, \cos(x), \sin(x), \ldots\}$. Furthermore the Fourier series of the function is a linear combination of $\{1, \cos(x), \sin(x), \ldots\}$. Thus, if the Fourier series converges to the function, it is a representation of f with respect to the trigonometric orthogonal basis $\{\cos(nx), \sin(nx)\}$ in the infinite space L^2.

2.4 Examples of Fourier series

With Lemma 2.5 giving the formula for finding the Fourier coefficients, we discuss two concrete examples of Fourier series. Besides, we make the claim on whether the partial sums must converge uniformly.

Example 2.1.

$$f(x) = \begin{cases} 1 & \textit{if } 0 < x < \pi \\ 0 & \textit{if } x = 0 \textit{ or } x = \pi \\ -1 & \textit{if } -\pi < x < 0 \end{cases}$$

First observe that $f(x)$ is an odd function, which means that integrating $f(x)$ over the domain $(-\pi.\pi]$ will be zero since postive and negative parts are canceled out.

Thus, by Lemma 2.5,

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx = 0.$$

Furthermore, since the product of an even function and and odd function is odd, we have

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)dx = 0.$$

Then, we need to evaluate the following integral to find b_n,

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)dx = \frac{2}{\pi}\int_{0}^{\pi} f(x)\sin(nx)dx,$$

since $f(x)\sin(nx)$ is even. Thus,

$$b_n = \begin{cases} 4/n\pi & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}$$

Combining those results together, we derive the Fourier series

$$f(x) = \frac{4}{\pi}\sum_{n=0}^{\infty}\frac{1}{2n+1}\sin((2n+1)x),$$

where the part $(2n+1)$ clarifies the case when n is odd.

As a side note, we discussed whether the partial sum of the Fourier series can converge to $f(x)$ uniformly. If it does converge uniformly, this function $f(x)$ may motivate us to conjecture that the uniform convergence is the necessary condition for the existence of Fourier Series. Unfortunately, the convergence of partial sums of $f(x)$ is not uniform on any interval containing 0. To see the reason, first notice that since any partial sum is continuous on any interval containing 0, then if uniform convergence holds, $f(x)$ must also be continuous on the same interval. However, $f(x)$ is defined to be discontinuous, which is the desired contradiction.

Example 2.2.

$$g(x) = |x|$$

Need to find the Fourier Series of $g(x)$.

First observe that $g(x)$ is an even function on $(-\pi, \pi]$. Then, since the product of an even and an odd function is odd, we have

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} g(x)\sin(nx)dx = 0.$$

Then, we proceed to evaluate a_0 and a_n as

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(x)dx = \frac{1}{\pi}\int_{0}^{\pi} xdx = \frac{\pi}{2},$$

and

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} g(x)\cos(nx)dx = \frac{2}{\pi}\int_{0}^{\pi} x\cos(nx) = \frac{2(\cos(n\pi)-1)}{n^2\pi}.$$

By the property of cos function, $(\cos(n\pi)-1) = 0$ when n is even. Therefore,

$$a_n = \begin{cases} -\frac{4}{n^2\pi} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}$$

Combining those results together, we derive the Fourier series

$$g(x) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}\cos((2n+1)x),$$

where the part $(2n+1)$ clarifies the case when n is odd.

Then, from the formula, we observe that the partial sums of the Fourier series of g converge. Notice that the term in the partial sum

$$\left|\frac{1}{(2n+1)^2}\cos((2n+1)x)\right| \leq \frac{1}{(2n+1)^2},$$

for each $n \in \mathbb{N}$ and $x \in (\pi, \pi]$. Moreover, the infinite series $\sum_{n=1}^{\infty} \frac{1}{(2n+1)^2}$ converges. Then, by applying the Weierstrass M-Test (Corollary 6.4.5, Abbott), partial sums of the Fourier series converge uniformly on $(\pi, \pi]$.

3 Convergence Theorems

In this section, we investigate Fourier's question of "which functions can be expressed as a trigonometric series" [3]. First, notice that there are many types of convergence of functions such as point-wise convergence, uniform convergence, and convergence in L^p spaces. The more general the type of convergence is, the more functions there are that we can express as a limit of trigonometric series.

In our following discussion, we will first consider the relatively weak pointwise convergence of functions of continuous functions. Then, we turn our attention to convergence of functions in the especially nice Hilbert space L^2 (an inner product space with a completed induced norm with respect to the space). To facilitate our discussion, we first define partial sums of Fourier series.

Definition 3.1. *For $f \in L^2[-\pi, \pi]$ the sequence $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ as defined in Lemma 2.5 are the Fourier coefficients for f. We define the Fourier series for f as,*

$$\frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k \cos(kx) + b_k \sin(kx)).$$

Let $s_n(x)$ denote the n-th partial sum of the Fourier series of f,

$$s_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n}(a_k \cos(kx) + b_k \sin(kx)).$$

3.1 Pointwise Convergence of Fourier Series

Theorem 3.1 (Abbott, [1])**.** *Let f(x) be continuous on $(-\pi, \pi]$, and let $s_n(x)$ as the nth partial sum of the Fourier Series as defined above. Let a_n and b_n be the coefficients defined in Lemma 2.5. Then,*

$$\lim_{n\to\infty} s_n(x) = f(x),$$

where the limit denotes the pointwise limit at any $x \in (-\pi, \pi]$ where $f'(x)$ exists.

The proof of the Theorem is omitted, but we will try to sketch the proof for the more powerful type of Fourier Series convergence theorem in the next subsection. Before turning our heads to the convergence in L^2, it is necessary to check which functions have a Fourier Series representation by applying Theorem 3.1. We will also discuss how Fourier series can be compared to the familiar power series. First, we notice that the functions in Theorem 3.1 are restricted to be continuous. Thus the function $g(x) = |x|$ mentioned in Example 2.2 has a Fourier Series representation. However, piece-wise continuous functions like the one we analyzed in Example 2.1 are not guaranteed to have a Fourier Series representation because the function is not continuous on the entire domain. But still, we may improve Theorem 3.1 to a more general version on pairwise continuous functions to include more examples. However, there are a lot of discontinuous functions which cannot be studied with Theorem 3.1. Thus, Fourier's claim that "There is no function $f(x)$ or part of a function, which cannot be expressed by a trigonometric series," is far from being proved.

Moreover, we also observe that the existence of a Fourier Series representation is far less robust than the power series. Theorem 6.5.1 and 6.5.2 in Abbott [1] together imply that a function has a power series representation on the interval $[-r, r]$ if $\sum_{n=0}^{\infty} a_n x^n$ converges at a point $x_0 = R$, where $0 < r < R$. Thus, to test whether the power series converges to a function on an interval, one has to use convergence tests at some x, while to test whether the Fourier series converges, one needs to check the property of the target function with respect to the type of convergence of interest. Therefore, the convergence of Fourier series is more powerful than the convergence of power series in the sense that the convergence is more versatile.

3.2 Convergence of Fourier Series in L^2

We want to prove that any $f \in L^2[-\pi, \pi]$ satisfies that it is "equal" to its Fourier series. In this particular context, we want the $s_n(x)$ to converge to $f(x)$ with respect to the L^2 norm. In particular we want to show,

$$\lim_{n\to\infty} \|f - s_n\|_2 = 0$$

which means,

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k \cos(kx) + b_k \sin(kx))$$

inside of this framework. Before providing a proof of this result, we will list some useful facts. We will not provide formal proofs of these facts, instead, we will supply some proof ideas.

Proposition 3.1 (Nelson, [2]). *Let $f \in L^2[-\pi, \pi]$. Then for each positive integer n,*

$$\|f - s_n\|_2^2 = \|f\|_2^2 - \left(\frac{\pi a_0^2}{2} + \pi \sum_{k=1}^{n}(a_k^2 + b_k^2)\right),$$

where $a_0, a_1, b_1, a_2, b_2, \ldots$ are the Fourier coefficients defined above and $s_n(x)$ is the n-th partial sum of the Fourier series for f.

The above fact is proven using $(f - s_n)^2 = f^2 - 2fs_n + (s_n)^2$ and thus, $\|f - s_n\|_2^2 = \int_{-\pi}^{\pi} f^2 - 2\int_{-\pi}^{\pi} fs_n + \int_{-\pi}^{\pi}(s_n)^2$. Note that most we know the value of the terms in the expansion of $\int_{-\pi}^{\pi}(s_n)^2$ by the lemmas in section 2.2 of this paper. After distributing f inside of the sum in s_n, we can note that the terms in $\int_{-\pi}^{\pi} fs_n$ resemble the coefficients found in Lemma 2.5.

Proposition 3.2 (Nelson, [2]). *Let $f \in L^2[-\pi, \pi]$. Let $a_0, a_1, b_1, a_2, b_2, \ldots$ be the Fourier coefficients for f. Then the series $\sum_{k=1}^{\infty}(a_k^2 + b_k^2)$ converges and*

$$\frac{\pi a_0^2}{2} + \pi \sum_{k=1}^{\infty}(a_k^2 + b_k^2) \leq \|f\|_2^2.$$

This result is in reality a corollary of the previous proposition. The central idea is that, $\|f - s_n\|_2^2 \geq 0$ so that,

$$\frac{\pi a_0^2}{2} + \pi \sum_{k=1}^{n}(a_k^2 + b_k^2) \leq \|f\|_2^2$$

for every n by Proposition 3.1. Then we can treat $\sum_{k=1}^{n}(a_k^2 + b_k^2)$ as a sequence and apply the algebraic limit theorem and the order limit theorem to conclude the aforementioned inequality.

Definition 3.2. *For a positive integer n, a trigonometric polynomial of degree n is a function T_n of the form,*

$$T_n(x) = A_0 + \sum_{k=1}^{n}(A_k \cos(kx) + B_k \sin(kx))$$

where $A_0, A_1, B_1, A_2, B_2, \ldots$ are real numbers.

Note that for a function $f \in L^2[-\pi, \pi]$, the n-th partial sum of its Fourier series is an example of a trigonometric series of degree n. The following result asserts that given $f \in L^2[-\pi, \pi]$, s_n is the trigonometric polynomial of degree n closest to f. Here, the notion of closeness is given by the L^2 norm.

Theorem 3.2 (Nelson, [2]). *Let $f \in L^2[-\pi, \pi]$ and let T_n be any trigonometric polynomial of degree n. Then,*

$$\|f - T_n\|_2 \geq \|f - s_n\|_2$$

where s_n is the n-th partial sum of Fourier series for f.

Since $\|f - T_n\|_2$ and $\|f - s_n\|_2$ are positive real numbers, the goal is to show that $\|f - T_n\|_2^2 - \|f - s_n\|_2^2 \geq 0$. For this, one can see that $\|f - T_n\|_2^2 = \int_{-\pi}^{\pi} f^2 - 2\int_{-\pi}^{\pi} fT_n + \int_{-\pi}^{\pi}(T_n)^2$. Then observe that we know the value of the terms in the expansion of $\int_{-\pi}^{\pi}(T_n)^2$ by the lemmas in section 2.2 of this paper. After distributing f inside of the sum in T_n, we can note that the terms in $\int_{-\pi}^{\pi} fT_n$ resemble the coefficients found in Lemma 2.5. After that, one can use Proposition 3.1 and algebraic manipulations to show that $\|f - T_n\|_2^2 - \|f - s_n\|_2^2 \geq 0$.

Theorem 3.3 (Nelson, [2])**.** *Let $p \geq 1$ and $f \in L^p[a,b]$. For every $\varepsilon > 0$ there exists a continuous function g defined on $[a,b]$ such that $\|f-g\|_p < \varepsilon$.*

The proof of this result is in Chapter 3.3 of our book, and it says that any function in $L^p[a,b]$ can be approximated by a continuous function g. Geometrically, this means that there is a continuous function g "near" f with respect to the L^p norm. In fact we can do better,

Corollary 3.3.1 (Nelson, [2])**.** *Let $f \in L^p[a,b]$ for $p \geq 1$. Given two real numbers A and B, and $\varepsilon > 0$, there is a continuous function g defined on $[a,b]$ with $g(a) = A$ and $g(b) = B$, such that $\|f-g\|_p < \varepsilon$..*

Theorem 3.4 (Nelson, [2])**.** *Let f be continuous on $[-\pi,\pi]$ with $f(-\pi) = f(\pi)$. For each positive integer n, define*

$$\sigma_n(x) = \frac{1}{n}\sum_{k=0}^{n-1} s_n(x)$$

where $s_n(x)$ is the n-th partial sum of the Fourier series for f. Then the sequence of functions $\sigma_n(x)$ converges uniformly to f on $[-\pi,\pi]$.

The proof of this result studies these functions,

Definition 3.3. *For a positive integer n, the n-th Dirichlet kernel is,*

$$D_n(t) = \frac{1}{2} + \sum_{k=1}^{n} \cos(kt).$$

By convention, $D_0(t) = \frac{1}{2}$. And the n-th Fejér kernel is

$$K_n(t) = \frac{1}{n}\sum_{k=0}^{n-1} D_n(x).$$

As a side note, $D_n(t)$ is an even periodic function with,

$$\int_0^{\pi} D_n(t)dt = \frac{\pi}{2},$$

and $K_n(t)$ is also periodic with,

$$\int_0^{\pi} K_n(t)dt = \frac{\pi}{2}.$$

Corollary 3.4.1 (Nelson, [2])**.** *Let f be continuous on $[-\pi,\pi]$ with $f(-\pi) = f(\pi)$. For each positive integer n, define*

$$\sigma_n(x) = \frac{1}{n}\sum_{k=0}^{n-1} s_n(x)$$

where $s_n(x)$ is the n-th partial sum of the Fourier series for f. Then,

$$\lim_{n\to\infty} \|\sigma_n - f\|_2 = 0.$$

The previous theorem asserts that $\sigma_n \to f$ uniformly, which means that $|\sigma_n - f|^2 \to 0$ uniformly. Thence, the previous result follows by Theorem 7.4.4 in Abbott [1].

Now we are ready to state our final result.

Theorem 3.5. *Let $f \in L^2[-\pi,\pi]$. Let $s_n(x)$ equal the n-th partial sum of the Fourier series for f. Then the sequence s_n converges to f with respect to the L^2 norm. In other words,*

$$\lim_{n\to\infty} \|s_n - f\|_2 = 0.$$

Proof. Let $\varepsilon > 0$. By corollary 3.2.1 there is a continuous function g, where $g(-\pi) = g(\pi)$ (we are free to pick both values, but here we are content with $g(-\pi)$ being equal to whatever $g(\pi)$ is) and,

$$\|f - g\|_2 < \frac{\varepsilon}{2}.$$

Let σ_n be the function defined in Theorem 3.3, but for g. That is,

$$\sigma_n(x) = \frac{1}{n}\sum_{k=0}^{n-1} h_k(x),$$

where h_k is the k-th partial sum of the Fourier series for g. Since g is continuous and $g(-\pi) = g(\pi)$, we can use Corollary 3.3.1 to state,

$$\lim_{n\to\infty} \|g - \sigma_n\|_2 = 0.$$

Thus there exists a positive integer N such that for $n \geq N$,

$$\|g - \sigma_n\|_2 < \frac{\varepsilon}{2}.$$

Thus, let $n \geq N$. Note that $\sigma_{n+1}(x)$ is a trigonometric polynomial of degree n, as the largest trigonometric polynomial in the sum in the definition of σ_{n+1} is $h_n(x)$. Thus by Theorem 3.1 and Minkowski's inequality,

$$\begin{aligned}
\|f - s_n\|_2 &\leq \|f - \sigma_{n+1}\|_2 \\
&= \|f - g + g - \sigma_{n+1}\|_2 \\
&\leq \|f - g\|_2 + \|g - \sigma_{n+1}\|_2 \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

■

Corollary 3.5.1. *Let $f \in L^2[-\pi, \pi]$. Let $a_0, a_1, b_1, a_2, b_2, \ldots$ be the Fourier coefficients for f. Then,*

$$\frac{\pi a_0^2}{2} + \pi\sum_{k=1}^{\infty}(a_k^2 + b_k^2) = \|f\|_2^2.$$

Proof. From Proposition 3.1, we have that for every n,

$$\|f - s_n\|_2^2 = \|f\|_2^2 - \left(\frac{\pi a_0^2}{2} + \pi\sum_{k=1}^{n}(a_k^2 + b_k^2)\right).$$

The result follows from taking the limit on both sides and using the previous theorem. ■

We end out discussion of Fourier series with the following remarks.

Theorem 3.1 is a weaker version of Theorem 3.5, because the former requires that the function f be continuous on $(-\pi, \pi]$. Note that the function f^2 is also continuous, and thus that it is Riemann and Lebesgue integrable on that interval. Then $f \in L^2(-\pi, \pi]$, and the result of Theorem 3.1 follows by applying Theorem 3.5. Then Theorem 3.5 implies Theorem 3.1.

On the other hand, recall that from our earlier remarks, that,

$$\alpha = \{1, \cos(x), \sin(x), \cos(2x), \sin(2x), \ldots\},$$

is a list of vectors in $L^2[-\pi, \pi]$ that are all "pairwise" orthogonal to each other. Furthermore, given a function $f \in L^2[-\pi, \pi]$ the coefficients a_i and b_i are the scalars that arise from projecting f onto each of the vectors in α (this is what we usually do in the finite dimensional case). Thus, with a Linear Algebra lens, we note that the Fourier series of f is the same as writing f as a linear combination of the vectors in α. Thus, Theorem 3.5 tells us that in fact, every $f \in L^2[-\pi, \pi]$ can be written as the linear combination of the vectors in α. This linear combination must be unique by Theorem 3.2, because any other trigonometric series that is not the Fourier series is further from f than the Fourier series. The vectors in α are linearly independent by virtue of being orthogonal, so that the linear algebra interpretation of Theorem 3.5 is that α forms a basis of $L^2[-\pi, \pi]$. This result is not as straight forward as it previously appeared to us, because infinite vector spaces do not necessarily have the same nice bases as finite dimensional vector spaces, even ones with inner products.

With Rafe we discussed the importance of this result being done in $L^2[\pi, \pi]$. The first thing that comes to mind when thinking about $L^2[\pi, \pi]$ is that it is the only $L^p[-\pi, \pi]$ space that is also a Hilbert space, but we looked in vain to see where, if at all, completeness was used in this proof.

Finally, we will describe a bird's-eye view of the proof of Theorem 3.5. This proof uses three main results: Theorem 3.2, Theorem 3.3 (disguised as Corollary 3.3.1), and Theorem 3.4 (disguised as Corollary 3.4.1). As for the two remaining results of this section, note that Proposition 3.1 is used to prove Theorem 3.2 and Proposition 3.2 is a weaker version of Corollary 3.5.1 (In a sense, Proposition 3.2 is a consolation prize if Theorem 3.5 were not true).

How do these three Theorems interact with each other to prove Theorem 3.5? The first key step is Theorem 3.4, as it claims that for a continuous g, there is a trigonometric series σ that converges absolutely to g. This result, in reality, is the only result in which we have directly prove the convergence of a trigonometric series, and it could be argued that here is were all of the work happens in the proof of Theorem 3.5 (there is a lot of work done to prove Theorem 3.3, as well).

Thus, Theorem 3.4 says that any function in $L^2[-\pi, \pi]$ can be approximated by a continuous function g, for which there is a trigonometric function that converges to g. Thus, using Theorem 3.2, we can bound the distance of f to s_n, by the distance between f and σ_{n+1}, which is in turn bounded by the approximation of f with g and the convergence of g to σ_{n+1}.

In a way, this proof says that we can show that f is the limit of its Fourier series, because we can approximate f to a periodic continuous function g, that in turn is a limit of a trigonometric series. Because the Fourier series cannot be further from f than the distance f is from g and the trigonometric series, f must be arbitrary close to its Fourier series.

References

[1] S. Abbott, Understanding Analysis. Springer-Verlag New York, 978-1-4939-2711-1, 2015

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[3] G. Fourier, Theorie Analytique de la Chaleur, F. Didot, 1822.