(CO)HOMOLOGY AND TOPOLOGICAL COMPLEXITY

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1. Preliminaries

1.1. **Homotopy Equivalence.** Given topological spaces X and Y, we are aided by a sense of equivalence which broadly captures the shape of these spaces. The notion of *homeomorphism* may be familiar, where two spaces are homeomorphic if there exists a continuous map from one to the other with a continuous inverse. In our case, we are interested in a broader sense of equivalence, and this is the notion of homotopy equivalence. To begin, we have the following definitions:

Definition 1.1. Two maps $f, g : X \to Y$ are *homotopic* if there exists a continuous map $F : X \times I \to Y$ such that $F(x, 0) = f(x)$, $F(x, 1) = g(x)$ for all $x \in X$. In this case, we write $f \simeq g$.

Definition 1.2. Two spaces X and Y are *homotopy equivalent* if there exist maps $f : X \to Y$ and $g : Y \to X$ such that fg and gf are homotopic to the identities on Y and X, respectively.

From these preliminary definitions, we can note that homeomorphism is a special case of homotopy equivalence: taking $g = f^{-1}$ (since f is a continuous map with continuous inverse), we have that the compositions fg and gf are equal (so homotopic) to the identity maps on their respective spaces.

Next, we can establish the following definition:

Definition 1.3. A space X is *contractible* if it is homotopy equivalent to a point.

As a couple of early examples, $\mathbb{R}^n$ is a contractible space, while S^n is not. In our work with topological complexity, we will see that many (if not all) spaces of interest are not contractible. In particular, it turns out that every contractible space has topological complexity of 1, and we will make all of these ideas precise in the next section.

1.2. **Algebraic Preliminaries.** Next, we will introduce the algebraic concepts of exact sequences and tensor products, both of which will ultimately prove useful in our work with homology and cohomology.

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Definition 1.4. A sequence of groups and homomorphisms

$$\cdots \longrightarrow A_{n+1} \xrightarrow{\alpha_{n+1}} A_n \xrightarrow{\alpha_n} A_{n-1} \xrightarrow{\alpha_{n-1}} \cdots$$

is *exact* if $\ker \alpha_n = \operatorname{im} \alpha_{n+1}$.

Beyond its definition, the exactness of a sequence can allow us to deduce a number of properties of groups and homomorphisms in between them.

Corollary 1.5. *Given groups A, B, C and group homomophisms $\phi : A \to B$ and $\psi : B \to C$, we have the following:*

1. *If $A \xrightarrow{\phi} B \longrightarrow 0$ is exact, then ϕ is surjective.*
2. *If $0 \longrightarrow A \xrightarrow{\phi} B$ is exact, then ϕ is injective.*
3. *If $0 \longrightarrow A \xrightarrow{\phi} B \longrightarrow 0$ is exact, then ϕ is an isomorphism.*
4. *If $0 \longrightarrow A \xrightarrow{\phi} B \xrightarrow{\psi} C \longrightarrow 0$ is exact, then $C \approx B/\ker\psi = B/\operatorname{im}\phi$*

In particular, the first two statements follow immediately from the definition, the third follows from the first two, and the fourth comes from the First Isomorphism Theorem.

Next, we give a definition of tensor product of abelian groups.

Definition 1.6. For abelian groups A, B, the *tensor product* $A \otimes B$ is generated by elements of the form $a \otimes b$, with $a \in A$ and $b \in B$, such that the following two properties hold:

1. $(a + a') \otimes b = a \otimes b + a' \otimes b$, and
2. $a \otimes (b + b') = a \otimes b + a \otimes b'$.

An extra properties $(na) \otimes b = n(a \otimes b) = a \otimes nb$ directly follows from property (1) and (2).

Theorem 1.7. *A bilinear map $\phi : A \times B \to C$ induces a homomorphism $\Phi : A \otimes B \to C$ given by $a \otimes b \mapsto \phi(a, b)$.*

Proof. It suffices to show that the induced relation respects the two properties of tensor product. To this end, observe that $\Phi((a + a') \otimes b) = \phi(a + a', b)$ (by the definition of Φ) $= \phi(a, b) + \phi(a', b)$ (by the property of a bilinear map) $= \Phi(a \otimes b) + \Phi(a' \otimes b)$ (by definition again). Similarly, we can show that $\Phi(a \otimes (b + b')) = \Phi(a \otimes b) + \Phi(a \otimes b')$. Therefore, Φ is the induced homomorphism by the definition $a \otimes b \mapsto \phi(a, b)$. $\square$

The theorem can be applied to the cup product for cohomology groups so that an induced homomorphism is guaranteed by the tensor product, which will be discussed in details soon.

1.3. **Homology and Cohomology.** Homology is a general way defined to associate groups to a topological space. One type of homology is simplicial homology. Although simplicial homology is restricted to CW-complexes, it is more computationally friendly than its generalization — singular cohomology. Here, we give a construction of homology groups for S^1.

Example 1.8. Here, we first construct the free abelian group generated by n-simplices of S^1. Note that a circle can be constructed as a 1-cell with two end points being equivalent. Therefore, $\Delta_0(X)$ is generated by the vertex, which we call x and $\Delta_1(X)$ is generated by the edge, which we call e ($e \mapsto 0$). For $n > 1$, $\Delta_n(X) = 0$. Therefore, maps $\partial : \Delta_n(X) \to \Delta_{n-1}(X)$ can be defined.

With everything labeled in the diagram, we can calculate the simplicial homology groups:

$$H_0^\Delta = \frac{\ker \partial_0}{\operatorname{im} \partial_1} = \frac{\Delta_0(X)}{\{0\}} \approx \mathbb{Z} \quad H_1^\Delta = \frac{\ker \partial_1}{\operatorname{im} \partial_2} = \frac{\Delta_1(X)}{\{0\}} \approx \mathbb{Z}.$$

$$\begin{tikzcd}
0 \arrow[r, "\partial_2"] & \Delta_1(X) \arrow[r, "\partial_1"] \arrow[d, "\approx"] & \Delta_0(X) \arrow[r, "\partial_0=0"] \arrow[d, "\approx"] & 0 \\
& \langle e \rangle & \langle x \rangle &
\end{tikzcd}$$

The first observation is that the zero homology group of circle is $\mathbb{Z}$, which agrees with the result that any path-connected space has $H_1 \approx \mathbb{Z}$. Secondly, the result can be generalized to any S^n as $H_0^\Delta \approx H_n^\Delta \approx \mathbb{Z}$, while other homology groups are trivial.

The construction of singular homology is similar to the simplicial homology. However, the free abelian groups $C_n(X)$ are generated by all singular n-simplicies in X. As a result, $C_n(X)$ are usually infinitely generated. Perhaps unsurprisingly, if X is a CW-complex, the singular homology groups are isomorphic to simplicial homology groups.

One significance of homology and cohomology groups is that they are invariants under homotopy equivalence. In other words, spaces that are homotopy equivalent have isomorphic homology and cohomology groups. Therefore, to study different classes of topological spaces, one can instead study the corresponding algebraic structures with more ease.

Theorem 1.9. *A homotopy equivalence $f : X \to Y$ induces an isomorphism $f_* : H_n(X) \to H_n(Y)$.*

Proof. The result follows from the result that two homotopic maps induces the same homology maps; i.e., if $f \simeq g : X \to Y$, then $f_* = g_* : H_n(X) \to H_n(Y)$. Considering the maps

$$H_n(X) \xrightarrow{f_*} H_n(Y) \xrightarrow{g_*} H_n(X) \xrightarrow{f_*} H_n(Y),$$

we have that $(gf)_* = \mathbb{1}_*$ and $(fg)_* = \mathbb{1}_*$ since, by definition of homotopy equivalence, fg and gf are homotopic to the identity maps on Y and X respectively. Having $(gf)_* = g_*f_* = \mathbb{1}_*$ implies that

f_* is injective, while $(fg)_* = f_* g_* = \mathbb{1}_*$ implies that f_* is surjective, so in all, f_* is an isomorphism. (As a note, the statement and proof for cohomology is similar.) $\square$

Next, cohomology is the "reverse" of homology as a way to associate a topological space with a group structure. While in the simplicial homology, we denote the free abelian groups in the chain complex as $C_n(X)$ generated by n-cells, in simplicial cohomology, we denote the dual groups in the cochain complex as $C^n(X)$. As an example, when the dual group is with respect to $\mathbb{Z}$, $C^n(X)$ equals to $\mathrm{Hom}(C_n(X), \mathbb{Z})$, which means homomorphisms from $C_n(X)$ to $\mathbb{Z}$. With well-defined coboundaries δ, the cohomology groups can be denoted as $H^n(X) = \ker \delta / \operatorname{im} \delta$.

One advantage of cohomology groups is that with its cup product, cohomology has a graded ring structure attatched to it.

Definition 1.10. Given cochains $\phi \in C^k(X;\mathbb{R})$ and $\psi \in C^l(X;\mathbb{R})$, then we define the *cup product* $\phi \smile \psi \in C^{k+l}(X;\mathbb{R})$ as

$$(\phi \smile \psi)(\sigma) = \phi(\sigma \mid_{[v_0,\ldots,v_k]}) \cdot \psi(\sigma \mid_{[v_k,\ldots,v_{k+l}]}),$$

where σ is a singular simplex from Δ^{k+l} to X.

With the definition of cup product, the map of cochains

$$C^k(X;\mathbb{R}) \times C^l(X;\mathbb{R}) \xrightarrow{\smile} C^{k+l}(X;\mathbb{R})$$

has well-defined coboundary and cycle. Thus, it induces the map of cohomology groups

$$H^k(X;\mathbb{R}) \times H^l(X;\mathbb{R}) \xrightarrow{\smile} H^{k+l}(X;\mathbb{R})$$

Although this induced map is only a function but not a ring homomorphism, with the definition of tensor product and the Theorem 1.7, the induced map can be modified to a ring homomorphism

$$H^k(X;\mathbb{R}) \otimes H^l(X;\mathbb{R}) \xrightarrow{\smile} H^{k+l}(X;\mathbb{R}).$$

Given this cup product, we can define the *zero-divisors cup length* of $H^*(X;\mathbf{k})$ to be the largest i such that there exist $\alpha_1, \ldots, \alpha_i \in \ker \smile$ such that $\alpha_1 \cdots \alpha_i \neq 0$ in $H^*(X;\mathbf{k}) \otimes H^*(X;\mathbf{k})$. As a note, since $H^*(X;R)$ is a graded ring, it is the case that the tensor product $H^*(X;R) \otimes H^*(X;R)$ is also a graded ring. In particular, the product in $H^*(X;R) \otimes H^*(X;R)$ is given by

$$(a \otimes b) \cdot (c \otimes d) = (-1)^{|b||c|} ac \otimes bd,$$

where $|b|$ and $|c|$ are the associated dimensions of b and r in $H^*(X;R)$. As we will show in the final section, this zero-divisors cup length provides a lower bound on the topological complexity of a given space X.

As one final tool in working with cohomology, we are aided by the following Künneth formula which provides a way of computing the cohomology of product spaces with tensor products:

Theorem 1.11. *If X and Y are finite cell complexes and $\mathbf{k}$ is a field, we have*

$$H^*(X;\mathbf{k}) \otimes H^*(Y;\mathbf{k}) \approx H^*(X \times Y;\mathbf{k}).$$

In this situation, we can then express the cup product as the composition

$$H^*(X) \otimes H^*(X) \xrightarrow{\approx} H^*(X \times X) \xrightarrow{\Delta^*} H^*(X)$$

where $\Delta : X \to X \times X$ is given by $x \mapsto (x, x)$.

2. Defining Topological Complexity

Topological complexity is an important notion related to the motion-planning problem that encodes how many different "rules" can describe the motion in a space. With a topological space X, motion planning involves taking a pair of configurations $(A, B) \in X \times X$ and producing a corresponding path $s(A, B)$ in the path space. The results of motion planning problems can be widely used in robotic programming in complicated settings with potential obstacles. By understanding the motion planning, the efficiency and accuracy of the programming can be practically improved.

Formally, the path space PX is defined as the collection of all the paths $\gamma : [0, 1] \to X$. Then, we define $\pi : PX \to X \times X$ as the map from the path to its endpoints; i.e., $\pi(\gamma) = (\gamma(0), \gamma(1))$. Therefore, to find a path from A to B, it's equivalent to find the section of π, which we call $s : X \times X \to PX$ that satisfies $\pi \circ s = \mathbb{1}_{X\times X}$.

The idea of topological complexity restricts the sections to be continuous, which benefits robot motion programmings to be accurate and effective. As a counterexample, if the section is not continuous, the jump in the programming may cause some systematic error. Continuity of sections can be naturally interpreted in metric spaces using the ϵ-δ definition as when the end points are moved to a small distance around its neighborhood, the choice of path also changes slightly. Note that the continuity can be rigorously defined using compact-open topology, which we will not discuss here.

With the interpretation of continuity of sections, we can define the topological complexity.

Definition 2.1. The *topological complexity* of a configuration space X, denoted as $\mathbf{TC}(X)$, is the minimal number of open sets $U_1, U_2, ..., U_{\mathbf{TC}(X)}$ in an open cover of $X \times X$ such that each U_i is paired with a continuous section $s_i : U_i \to PX$.

As implied by the statement of the definition, not every space has a single continuous section. Indeed, one can observe that the existence of such continuous section on the entire space is equivalent to the space being contractible.

Theorem 2.2. *A continuous motion planning $s : X \times X \to PX$ exists if and only if the configuration space X is contractible.*

Proof. ($\Rightarrow$) Assume the existence of such continuous section, which we call $s : X \times X \to PX$. Then fix an arbitrary point $A_0 \in X$. Then, the homotopy h_t from A_0 to X can be explicitly defined as

$$h_t(B) = s(A_0, B)(t),$$

where $B \in X$. Note that for h_t, $h_t(A) = A, h_0(B) = A_0$, and $h_1(B) = B$. Therefore, h_t implies a contraction of a space X into a point A_0.

($\Leftarrow$) Let's call the homotopy as h_t such that $h_0(A) = A$, and $h_1(A) = A_0$, where $A, A_0 \in X$. Then, we try to construct a continuous section.

For any $(A, B) \in X \times X$, define the motion planning as the composition of the path $h_t(A)$ and the inverse of $h_t(B)$ (we may reparametrize the path so that the motion planning is at constant speed from $t = 0$ to $t = 1$). Here, such motion planning is continuous since it is inherited from the continuous homotopy. Intuitively, the motion planning first moves A to the point of contraction A_0, and then moves to B. $\square$

As a final elementary result in topological complexity, we can prove that the topological complexity of a space X is an invariant quantity under homotopy equivalence.

Theorem 2.3. $\mathbf{TC}(X)$ *is invariant under homotopy equivalence.*

Proof. Suppose that there exist continuous maps $f : X \to Y$ and $g : Y \to X$ such that $fg \simeq \mathbb{1}_Y$, the identity map on Y. Given that $fg \simeq \mathbb{1}_Y$, let $h_t : Y \to Y$ be a homotopy such that $h_0 = \mathbb{1}_Y$ and $h_1 = f \circ g$.

Here, we want to show that $\mathbf{TC}(Y) \leq \mathbf{TC}(X)$. Let $U \subset X \times X$ be an open subset such that there exists a continuous motion planning $s : U \to PX$ over U. Define $V = (g \times g)^{-1}(U) \subseteq Y \times Y$ (where $(g \times g)^{-1}(x_1, x_2) = (g^{-1}(x_1), g^{-1}(x_2))$ for any $(x_1, x_2) \in X \times X$). From this information, we can explicitly construct a continuous motion planning $\sigma : V \to PY$ over V. For $(y_1, y_2) \in V$

and $t \in [0,1]$, set

$$\sigma(y_1, y_2)(t) = \begin{cases} h_{3t}(y_1), & \text{for } 0 \le t \le \frac{1}{3}, \\ f(s(g(y_1), g(y_2))(3t-1), & \text{for } \frac{1}{3} \le t \le \frac{2}{3}, \\ h_{3(1-t)}(y_2), & \text{for } \frac{2}{3} \le t \le 1. \end{cases}$$

This map σ first takes $y_1 \in Y$ to $f(g(y_1)) \in Y$ under the homotopy h_t. Next, σ equals the image under f of the path from $g(y_1)$ to $g(y_2)$ (as points in X) provided by the continuous section s. Finally, σ takes $f(g(y_2))$ back to y_2 under the homotopy h_t, now in reverse.

If $\mathbf{TC}(X) = k$, then there exists a minimal open cover $U_1 \cup \ldots U_k = X \times X$ with continuous motion planning over each U_i. Applying the above construction, we then obtain an open cover $V_1 \cup \cdots \cup V_k$ of $Y \times Y$ with continuous motion planning over each V_i. Therefore, we have that $\mathbf{TC}(Y) \le \mathbf{TC}(X)$.

If X and Y are homotopy equivalent, we have that there exist maps $f : X \to Y$ and $g : Y \to X$ such that $fg \simeq \mathbb{1}_Y$ and $gf \simeq \mathbb{1}_X$. By the above result, it follows that $\mathbf{TC}(Y) \le \mathbf{TC}(X)$ and $\mathbf{TC}(X) \le \mathbf{TC}(Y)$, so $\mathbf{TC}(X) = \mathbf{TC}(Y)$, concluding the proof. □

3. Bounds on Topological Complexity

With these preliminary results of topological complexity established, we can introduce the following bounds on the topological complexity of a space: a cohomological lower bound and a dimensional upper bound.

3.1. **A Lower Bound in Cohomology.**

Theorem 3.1. *If* $\mathbf{k}$ *is a field, then* $\mathbf{TC}(X)$ *is greater than the zero-divisors-cup-length of* $H^*(X; \mathbf{k})$.

Proof. Since $\mathbf{k}$ is a field, we have by Theorem 1.11 that $H^*(X) \otimes H^*(X) \approx H^*(X \times X)$, and the cup product is given by the composition

$$H^*(X) \otimes H^*(X) \xrightarrow{\approx} H^*(X \times X) \xrightarrow{\Delta^*} H^*(X)$$

where $\Delta : X \to X \times X$ is given by $x \mapsto (x, x)$. Therefore, $Z(X) \approx \ker(\Delta^*)$, so we can treat each α_i as an element $\beta_i \in \ker(\Delta^*)$, and showing $\alpha_1 \alpha_2 \cdots \alpha_t = 0$ in $H^*(X) \otimes H^*(X)$ is equivalent to showing $\beta_1 \smile \cdots \smile \beta_t = 0$ in $H^*(X \times X)$. In other words, we want to show $\beta_1 \smile \cdots \smile \beta_t$ is in the kernel of the iterated cup product

$$\smile : H^*(X \times X) \otimes \cdots \otimes H^*(X \times X) \to H^*(X \times X).$$

To do so, we can let $a : X \to PX$ be the map that sends x to the constant path at x. Note that there exists a map $b : PX \to X$ that sends a path p in PX to $p(0) \in X$. Considering b as the composition of the map π (as introduced in the previous section) with the projection map $P : X \times X \to X$ defined by $(x, y) \mapsto x$, we can note that b is indeed continuous as a composition of continuous maps. Furthermore, we have that $ba = \mathbb{1}_X$ by definition, and we can note that $ab \simeq \mathbb{1}_{PX}$ by considering the homotopy $f_t : PX \to PX$ which takes any path $p : [0, 1] \to X$ to the path p' such that $p'(s) = p(s)$ for all $s \le t$, and $p'(s) = p(t)$ for all $s > t$. In particular, we have that $f_0 = ab$ and $f_1 = \mathbb{1}_{PX}$. Therefore, a is indeed a homotopy equivalence, so the induced homomorphism a^* is an isomorphism and we have the following commutative diagrams:

$$\begin{tikzcd} X \arrow[rr, "a"] \arrow[dr, "\Delta"'] & & PX \arrow[dl, "\pi"] \\ & X \times X & \end{tikzcd} \qquad \begin{tikzcd} H^*(X) & & H^*(PX) \arrow[ll, "a^*"'] \\ & H^*(X \times X) \arrow[ul, "\Delta^*"] \arrow[ur, "\pi^*"'] & \end{tikzcd}$$

Since a^* is an isomorphism, we have that $\ker(\Delta^*) = \ker(\pi^*)$.

Let $\mathbf{TC}(X) = t$, so there exist open sets $U_1, \ldots, U_t$ which cover $X \times X$ and continuous sections $s_i : U_i \to PX$. This gives the following commutative diagrams for each i (where j_i and k_i are inclusions):

$$\begin{tikzcd} \pi^{-1}(U_i) \arrow[r, hook, "j_i"] \arrow[d, shift left, "\pi"] & PX \arrow[d, "\pi"] \\ U_i \arrow[u, bend left, "s_i"] \arrow[r, hook, "k_i"] & X \times X \end{tikzcd} \qquad \begin{tikzcd} H^*(\pi^{-1}(U_i)) \arrow[d, bend left, "s_i^*"] & H^*(PX) \arrow[l, "j_i^*"'] \\ H^*(U_i) \arrow[u, "\pi^*"] & H^*(X \times X) \arrow[l, "k_i^*"'] \arrow[u, "\pi^*"'] \end{tikzcd}$$

Here, we can note that π^* is injective. To see this, note that $\pi s = \mathbb{1}_{U_i}$, so

$$(\pi s)^* = s^* \pi^* = \mathbb{1}_{U_i}^*.$$

In particular, we have that if $\pi^*(x) = \pi^*(y)$, then $s^*\pi^*(x) = x = y = s^*\pi^*(y)$, so π^* is indeed injective. By the commutativity of the right diagram above, we then have that $j_i^*\pi^* = \pi^* k_i^*$. If $\beta \in \ker(\pi^*)$, then we have that

$$0 = j_i^*\pi^*(\beta) = \pi^* k_i^*(\beta).$$

Since π^* is injective, we have that $\pi^* k_i^*(\beta) = 0$ implies that $k_i^*(\beta) = 0$, so $\beta \in \ker k_i^*$.

Next, for each i, we have a long exact sequence (as in Theorem 2.16 of Hatcher's *Algebraic Topology*)

$$\cdots \longleftarrow H^*(U_i) \xleftarrow[k_i^*]{} H^*(X \times X) \xleftarrow[q_i^*]{} H^*(X \times X, U_i) \longleftarrow \cdots$$

with the same map k_i^* as above. By the exactness of this sequence, we have that $\ker k_i^* = \operatorname{im} q_i^*$. Since $\beta \in \ker(\pi^*)$, we have that $\beta \in \ker k_i^* = \operatorname{im} q_i^*$. By the definition of the image, this then implies that there exists some $\widetilde{\beta} \in H^*(X \times X, U_i)$ such that $q_i^*(\widetilde{\beta}) = \beta$.

Now, we have the following commutative diagram:

$$\begin{tikzcd} H^*(X \times X, U_1) \otimes \cdots \otimes H^*(X \times X, U_t) \arrow[r, "\smile"] \arrow[d, "q_1^* \otimes \cdots \otimes q_t^*"] & H^*(X \times X, U_1 \cup \cdots \cup U_t) \arrow[d, "q^*"] \\ H^*(X \times X) \otimes \cdots \otimes H^*(X \times X) \arrow[r, "\smile"] & H^*(X \times X) \end{tikzcd}$$

As in the definition of topological complexity, we have that

$$X \times X = U_1 \cup \cdots \cup U_t,$$

so we have that

$$H^*(X \times X, U_1 \cup \cdots \cup U_t) = H^*(X \times X, X \times X) = 0.$$

Now, consider $\beta_1 \otimes \cdots \otimes \beta_t \in H^*(X \times X) \otimes \cdots \otimes H^*(X \times X)$. We can note from the previous paragraph that

$$(q_1^* \otimes \cdots \otimes q_t^*)(\widetilde{\beta}_1 \otimes \cdots \otimes \widetilde{\beta}_t) = \beta_1 \otimes \cdots \otimes \beta_t$$

for some $\widetilde{\beta}_1 \otimes \cdots \otimes \widetilde{\beta}_t \in H^*(X \times X, U_1) \otimes \cdots \otimes H^*(X \times X, U_t)$. In particular, we now have that the cup product

$$\smile : H^*(X \times X, U_1) \otimes \cdots \otimes H^*(X \times X, U_t) \to H^*(X \times X, U_1 \cup \cdots \cup U_t) = 0$$

sends every element of $H^*(X \times X, U_1) \otimes \cdots \otimes H^*(X \times X, U_t)$ to 0. Therefore, we have that

$$q^* \circ \smile (\widetilde{\beta}_1 \otimes \cdots \otimes \widetilde{\beta}_t) = 0$$

for all $\widetilde{\beta}_1 \otimes \cdots \otimes \widetilde{\beta}_t \in H^*(X \times X, U_1) \otimes \cdots \otimes H^*(X \times X, U_t)$. By the commutativity of the above diagram, we have that $\smile \circ (q_1^* \otimes \cdots \otimes q_t^*) = q^* \circ \smile : H^*(X \times X, U_1) \otimes \cdots \otimes H^*(X \times X, U_t) \to H^*(X \times X)$. Therefore, we have that

$$0 = \smile \circ (q_1^* \otimes \cdots \otimes q_t^*)(\widetilde{\beta}_1 \otimes \cdots \otimes \widetilde{\beta}_t) = \smile (\beta_1 \otimes \cdots \otimes \beta_t),$$

so $\beta_1 \otimes \cdots \otimes \beta_t \in \ker \smile$.

We now have that $\beta_1 \smile \cdots \smile \beta_t = 0$ in $H^*(X \times X)$, so equivalently, we have that $\alpha_1 \cdots \alpha_t = 0$. Therefore, we have that the zero-divisors-cup-length of $H^*(X; \mathbf{k})$ is less than t, the topological complexity of X, concluding the proof. $\square$

3.2. **Topological Complexity of Spheres.** As an sample application of this cohomological lower bound, we can consider the topological complexity of spheres S^n. We have that the cohomology groups of the sphere are

$$H^k(S^n;\mathbb{Q}) = \begin{cases} \mathbb{Q} & \text{if } k = 0, n \\ 0 & \text{otherwise.} \end{cases}$$

Let u be the generator for $H^n(S^n;\mathbb{Q})$ and let 1 be the generator of $H^0(S^n;\mathbb{Q})$. If we let $a = 1 \otimes u - u \otimes 1 \in H^*(S^n;\mathbb{Q}) \otimes H^*(S^n;\mathbb{Q})$, we can note that $\smile(a) = 1 \cdot u - u \cdot 1 = 0$, so $a \in \ker \smile$.

Next, we can note that

$$\begin{aligned} (u \otimes 1 - 1 \otimes u)(u \otimes 1 - 1 \otimes u) &= (-1)^{0 \cdot n}(u \smile u \otimes 1 \smile 1) \\ &\quad - (-1)^0(u \smile 1 \otimes 1 \smile u) \\ &\quad - (-1)^{n^2}(1 \smile u \otimes u \smile 1) \\ &\quad + (-1)^0(1 \smile 1 \otimes u \smile u) \\ &= -(u \smile 1 \otimes 1 \smile u) - (-1)^{n^2}(1 \smile u \otimes u \smile 1) \\ &= -(u \smile 1 \otimes 1 \smile u)(1 + (-1)^{n^2}) \\ &= \begin{cases} -2(u \otimes u) & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases}. \end{aligned}$$

Therefore, we have that the zero-divisors cup length of $H^*(S^n;\mathbb{Q})$ is at least one if n is odd and at least two if n is even. Applying the previous theorem, we then have that

$$\mathbf{TC}(S^n) \geq \begin{cases} 2 & \text{if } n \text{ is even} \\ 3 & \text{if } n \text{ is odd} \end{cases}.$$

From here, we can exhibit explicit motion planning rules to show equality, that is,

$$\mathbf{TC}(S^n) = \begin{cases} 2 & \text{if } n \text{ is even} \\ 3 & \text{if } n \text{ is odd} \end{cases}.$$

First, let $U_1 = \{(x, y) \mid x \neq -y\} \subset S^n \times S^n$ and define a continuous section $s_1 : U_1 \to PS^n$ which maps (x, y) to the shortest path from x to y.

Next, if n is odd, then by the Hairy Ball Theorem, there exists a nonzero tangent vector field on S^n. Take $U_2 = \{(x, y) \mid x \neq y\} \subset S^n \times S^n$, and define a continuous section $s_2 : U_2 \to PS^n$ which sends takes the point x to $-x$ along the arc in the direction of the tangent vector field, and then takes the shortest path from $-x$ to y. We now have that $\mathbf{TC}(S^n) \leq 2$ when n is odd, so together with our cohomological lower bound, we have that $\mathbf{TC}(S^n) = 2$ when n is odd.

If n is even, then we no longer have a nonzero tangent vector field on S^n, but we do have tangent vector fields which are nonzero except at one point, say, $x_0 \in S^n$. Define $U_2' = \{(x, y) \mid x \neq y,\ x \neq x_0\}$, and similarly define a continuous section $s_2' : U_2' \to PS^n$ which sends takes the point x to $-x$ along the arc in the direction of the tangent vector field, and then takes the shortest path from $-x$ to y.

Similarly, we have a tangent vector fields which are nonzero except at one point $-x_0 \in S^n$. Define $U_3' = \{(x, y) \mid x \neq y,\ x \neq -x_0\}$, and define a continuous section $s_3' : U_3' \to PS^n$ which sends takes the point x to $-x$ along the arc in the direction of the tangent vector field, and then takes the shortest path from $-x$ to y. We now have that $\mathbf{TC}(S^n) \leq 3$ when n is even, so together with our cohomological lower bound, we have that $\mathbf{TC}(S^n) = 3$ when n is even, as desired.

3.3. **An Upper Bound in Dimension.** To conclude, we will provide an upper bound of the topological complexity of a CW-complex X based on its dimension. In doing so, we are aided by the following lemma:

Lemma 3.2. *Given maps $f : A \to Y$ and $g : B \to Y$, if $\overline{A} \cap B = A \cap \overline{B} = \emptyset$, then the function $h : A \cup B \to Y$ defined by*

$$h(x) = \begin{cases} f(x) & \text{if } x \in A \\ g(x) & \text{if } x \in B \end{cases}$$

is continuous.

Proof. Let U be an open set in Y. Here, we want to show that $h^{-1}(U) = f^{-1}(U) \cup g^{-1}(U)$ is open in $A \cup B$.

First of all, we can note that $f^{-1}(U)$ is open in the subspace topology of A since f is a continuous map. Therefore, there exists an open set V of $A \cup B$ such that $f^{-1}(U) = A \cap V$. Here, we can note that since the closure $\overline{B}$ is closed in $A \cup B$, it follows that the complement $(\overline{B})^c$ is open in $A \cup B$. Furthermore, we have that $A \cap \overline{B}$ is empty, so we have that $A = (\overline{B})^c$ is an open set in $A \cup B$. Therefore, as the intersection of two open sets in $A \cup B$, we have that $f^{-1}(U) = A \cap V$ is itself an open set in $A \cup B$.

A similar argument provides that $g^{-1}(U)$ is also an open set in $A \cup B$, so we have that $h^{-1}(U) = f^{-1}(U) \cup g^{-1}(U)$ is open in $A \cup B$. Therefore, $h : A \cup B \to Y$ is indeed continuous. $\square$

Theorem 3.3. *If X is a connected CW complex of dimension n, then $\mathbf{TC}(X) \leq 2n + 1$.*

Proof. For $i = 0, 1, \ldots, n$, let $A_i = X_i - X_{i-1}$, so that A_i is the disjoint union of the open i-cells of X. Then, for $i = 0, 1, \ldots, 2n$, define $G_i \subset X \times X$ by

$$G_i = \bigcup_{r+s=i} A_r \times A_s.$$

Suppose e_r and e_s are open r-and s-cells of X (so that e_r and e_s are homeomorphic to open discs of dimensions r and s), with $r + s = i$ (so that $e_r \times e_s \subset G_i$). Fix points $x_0 \in e_r$ and $y_0 \in e_s$, and also fix some path γ (in X) from x_0 to y_0. Then, given any point $(x, y) \in e_r \times e_s$, let $s_i(x, y)$ be the path which takes the shortest path from x to x_0 (taken at constant speed), followed by γ, followed by the shortest path from y_0 to y (also at constant speed). This defines a continuous section s_i on $e_s \times e_r$ (which is continuous since the shortest paths from x to x_0 and from y_0 to y vary continuously with x and y).

From here, we can note that each each G_i is a union of such products of cells $e_r \times e_s$ satisfying $r + s = i$. We can apply the previous lemma to extend the continuous sections on each $e_r \times e_s$ to one continuous section on G_i. To do so, it suffices to prove that

$$\overline{e_r \times e_s} \cap e_p \times e_q = \emptyset,$$

whenever $r + s = p + q = i$.

Suppose that $\overline{e_r} \cap e_p \neq \emptyset$. In this case, if $p = r$, then it follows that $e_r = e_p$. This also implies that $s = q$, but by our setup, e_s and e_q must be different open cells of X, which implies an empty intersection $\overline{e_s} \cap e_q$ and hence an empty intersection $\overline{e_r \times e_s} \cap e_p \times e_q$. Next, when $r \neq p$, with the loss of generality, we assume $r > p$. Then, it follows that $s < q$, which in turn implies $\overline{e_s} \cap e_q = \emptyset$ (by considering the CW-complex structure). Thus, the intersection $\overline{e_r \times e_s} \cap e_p \times e_q$ must again be empty.

Repeatedly applying Lemma 3.2 then provides a continuous section defined on each of the $2n+1$ sets G_i, and since

$$X \times X = \bigcup_i G_i,$$

we have that $\mathbf{TC}(X) \leq 2n + 1$ (as the minimal number of such subsets), concluding the proof. $\square$

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