BROWN REPRESENTABILITY AND ITS VARIANTS IN CONTEXT

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ABSTRACT. Brown Representability is a fundamental theorem connecting the generalized cohomology theory and homotopy theory. Furthermore, Adams' variant theorems lift a countability assumption in Brown's work and create a one-to-one correspondence between generalized cohomology theories and $\Omega-$Spectra. Nowadays, given the categorical significance of Brown's proof, analogous representability theorems have been proved in more general contexts, like triangulated categories. In this paper, we develop and prove Adams' variant of Brown representability theorem and Neeman's representability theorem in a similar fashion to hint at a general theme of showing representability.

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1. INTRODUCTION

In 1945, Eilenberg and Steenrod axiomatized (co)homology theory including the dimension axiom. On the homotopy side, Eilenberg-MacLane spaces can be constructed such that the only nontrivial homotopy group is concentrated at a specific index. As we know, given a fixed coefficient group, there is a bijection between the homotopy classes of maps into Eilenberg-MacLane spaces and ordinary cohomology functors (for details, see REU paper [11]).

It was later realized that more powerful cohomology theories can be formed such as K-theory and complex cobordism without the dimension axiom. Thus, general (co)homology theories were introduced. Naturally, we want a homotopical description of general (co)homology theories. While it is well known that spectra represent general (co)homology theories (see Chapter 22 in Concise [6] for details),

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Brown's representability theorems in his papers [2] and [3] give the converse direction. However, one has to assume a countability condition on the cohomology. This assumption is unsatisfying since Brown's original representability theorems cannot imply the construction of Eilenberg-MacLane spaces when the coefficient group is $\mathbb{R}$ or $\mathbb{C}$.

In this paper, we first prove Adams' variant of representability Theorems, which solves the problem mentioned above. The trade-off is to assume that the given functor takes values in the category of groups, which is a stronger assumption but usually does not matter in practice. Crucially, Adams lifted the countability assumption to complete the bijection between spectra and general co(homology) theories commonly referred in the literature. This bijection is helpful when defining a cohomology theory abstractly. Moreover, the bijection can be extended in situations like equivariant homotopy theory, when some extra structure on cohomology theories is assumed.

Secondly, Brown representability is far more categorical than topological. We can generally define what it means for a contravariant functor H, from some category $\mathscr{C}$ to the category of sets $\mathscr{S}$, to be representable.

Definition 1.1. H is said to be *representable* if

$$\phi : [-, X] := \mathscr{C}(-, X) \approx H(-)$$

is a natural isomorphism for some fixed $X \in \mathscr{C}$. In this paper, we always use $[-, X]_{\mathscr{C}}$ or $[-, X]$ (if the category is clear) to denote $\mathscr{C}(-, X)$ unless stated otherwise.

In more general contexts like triangulated categories, a representability theorem can also be proved using the same strategy. We will highlight the common theme of representability in different contexts.

1.1. **Preliminaries.** Let's start with some necessary conditions on representable functors in the topological context and hope they are sufficient. If so, we will make them our axioms.

Notation 1.2. In this paper, all CW-complexes are assumed to be connected unless I say otherwise. Let CW be the category of CW-complexes with base-point and morphisms are homotopy classes of base-point preserving maps. Let $\widetilde{CW}$ be the subcategory of finite CW-complexes.

Axiom 1.3 (The Wedge Axiom)**.** Let $X = \bigvee_\alpha X_\alpha$ be the wedge-sum of arbitrary $X_\alpha \in CW$ with injections $i_\alpha : X_\alpha \to X$. Then, we have induced maps $i_\alpha^* = H(i_\alpha) : H(X) \to H(X_\alpha)$. Let the following function satisfy that the projection on the αth component is i_α^*:

$$\theta : H(X) \to \prod_\alpha H(X_\alpha).$$

The axiom says that θ is an isomorphism of sets in $\mathscr{S}$.

Remark 1.4. We note that the wedge sum is the coproduct in CW. Thus, Axiom 1.3 says that H sends coproducts in CW to products in $\mathscr{S}$.

Axiom 1.5 (The Mayer-Vietoris Axiom)**.** Let $X, Y \in CW$ and consider the following diagram. We let $x \in H(X), y \in H(Y)$ such that $ax = by$. The axiom says that there exists $z \in H(X \cup Y)$ such that $cz = x$ and $dz = y$.

$$\begin{array}{ccc} H(X \cap Y) & \xleftarrow{a} & H(X) \\ {\scriptstyle b}\uparrow & & \uparrow{\scriptstyle c} \\ H(Y) & \xleftarrow{d} & H(X \cup Y) \end{array}$$

Remark 1.6. Axiom 1.5 says that after applying H, we can find a "weak pullback" such that elements on the upper left come from the lower right but the process is not necessarily unique.

Remark 1.7. Note that H^q on the $q-th$ level of a general cohomology theory satisfies both the Wedge Axiom and the Mayer-Vietoris Axiom. The Wedge Sum Axiom is built into the axioms of general cohomology theories and the Mayer-Vietoris Axiom can be deduced from general Mayer-Vietoris exact sequences. Therefore, it is reasonable to assume those two axioms.

Next, we give a construction of the framework of Brown representability.

Construction 1.8. For a fixed $Y \in CW$, and $u \in H(Y)$, we have the following natural transformation $T_u : [-, Y] \to H(-)$ defined by

$$T_u(f) = H(f)u, \text{ where } f \in [X, Y].$$

Moreover, by Yoneda's Lemma, we have an essential 1-1 correspondence:

$$H(Y) \approx \text{ Nat Trans}([-, Y], H(-)).$$

Therefore, Brown's first Representability theorem essentially gives the choice of a specific $Y \in CW$ and $u \in H(Y)$ so that T_u is a natural isomorphism. We state the first version of Brown Representability theorem.

Theorem 1.9 (Brown [3])**.** *If $H : CW^{op} \to \mathscr{S}$ satisfies the Wedge Sum Axiom and the Mayer-Vietoris Axiom, then H is representable with representative (Y, u). Moreover, Y is unique up to weak homotopy equivalences.*

We immediately see that Brown Representability theorem is a generalization of Eilenberg-MacLane spaces. For n-sphere S^n, we note that

$$\pi_n(Y) = [S^n, Y] \approx H(S^n).$$

Since $H(S^n)$ is the coefficient group when $n = q$ and zero otherwise, the theorem gives the construction of Eilenberg-Maclane spaces without the obstruction theory.

However, in practice, we often define a cohomology theory only on finite CW-complexes and extend it to CW in the natural way. Thus, for a representability theorem to be useful, it needs to be an "extended" version. We first make the idea of "extension" precise.

Definition 1.10. Let H be a functor from $\widetilde{CW}$, the subcategory of finite CW complexes, to the category of sets. We define the *extension of* H into CW by

$$\widehat{H}(X) := \varprojlim_{\alpha} H(X_\alpha),$$

where X_α runs over all finite subcomplexes of X. Observe that if X is a finite CW complex, $\widehat{H}(X) = H(X)$ trivially.

Then, we state Brown's second theorem of representability.

Theorem 1.11 (Brown [3]). *If $H : \widetilde{CW}^{op} \to \mathscr{S}$ satisfies the Wedge Sum Axiom and the Mayer-Vietoris Axiom, by assuming $H(K)$ countable for all $K \in \widetilde{CW}^{op}$, we can extend H to $\widehat{H}$ defined on CW so that $\widehat{H}$ is representable.*

2. Adams' Variant of Brown Representability

In this section, we prove Adams' variant representability theorem, which differs from Brown's version in two key aspects: 1) Adams assumes the functor H to take values in the category of groups and homomorphisms, denoted as Gp; and 2) $H(K)$ is not necessarily countable for $K \in \widetilde{CW}$.

We state and prove the following two representability theorems of Adams in [1].

Theorem 2.1 (Adams [1]). *Let $H : \widetilde{CW}^{op} \to Gp$ satisfy the Wedge Axiom and the Mayer-Vietoris Axiom. Then, there is a CW-complex Y with natural isomorphisms*

$$T : [S^n, Y] \approx H(S^n)$$

for all n-spheres $S^n \in \mathcal{S}$, where $\mathcal{S}$ denotes the set of all n-spheres.

Before stating the extended representability theorem, we define a notion of "almost homotopy."

Definition 2.2. We say that two maps $X \to Y$ are *almost homotopic* (called "weak homotopy" by Adams in [1]), denoted as $f \sim_a g$, if $fh \sim gh$ for every map h from any finite complex K to X.

It can be easily checked that almost homotopy is an equivalence relation. Therefore, we can write $[X, Y]_a$ to denote the set of almost homotopy classes of maps from X to Y. We remark that when X is finite, $[X, Y]_a = [X, Y]$.

We state the following important theorem of Adams' and prove it in the rest of the section.

Theorem 2.3 (Adams [1]). *(i) There exists a unique natural transformation*

$$\widehat{T} : [X, Y]_a \to \widehat{H}(X)$$

such that the $T = \widehat{T}$ on finite CW-complexes K.

(ii) $\widehat{T}$ is a natural isomorphism of sets for all $X \in CW$.

Adams' Variant of Brown Representability has significance both formally and practically. From a formal standpoint, instead of adopting Brown's strategy of first deriving the representability within the subcategory of finite CW-complexes and then extending the result to CW, Adams exploits the property of $\widehat{H}$ defined in Definition 1.10. Namely, Adams shows that $\widehat{H}$ satisfies the Wedge and the Mayers-Vietories axiom to some extent. Then, Brown's proof of Theorem 1.9 can be replicated in the case of $\widehat{H}$ to obtain the desired result.

Practically, Adams' Theorem affirmatively says that all generalized cohomology theories are representable, regardless of the coefficient group. Also, through S-duality by Spanier in [9], Adams' theorem also gives the general homology representabilty that is a stronger than Whitehead's result in [10].

2.1. **General Inverse Limit.** Towards proving Adams' two theorems, we first need a new tool. Instead of using the inverse limit, we try to develop a "suitably nice" category structure to form a general notion of inverse limit. Specifically, we impose the following two conditions on a category $\mathscr{C}$:

1) for any objects $X, Y \in \mathscr{C}$, there is at most one morphism $f : X \to Y$ in $\mathscr{C}$,
2) and for any objects $X, Y \in \mathscr{C}$, we can pull them back to an object $Z \in \mathscr{C}$; i.e., there are morphisms $Z \to X$ and $Z \to Y$.

In this subsection, we always assume $\mathscr{C}$ to satisfy the above two conditions.

Definition 2.4. We define the *general inverse limit relative to* $\mathscr{C}$ to be the set $\varprojlim \mathscr{C}$ in which an element is a function e that assigns to each object $X \in \mathscr{C}$ an element e_X in X such that the following condition is satisfied:

$$\text{for each morphism } f : X \to Y, \text{we have } fe_X = e_Y.$$

Remark 2.5. Notice that if $\mathscr{C}$ is an inverse system, the general inverse limit is equivalent to the usual inverse limit. However, the major difference is that for our category $\mathscr{C}$, we do not require the existence of $f : X \to Y$ and $g : Y \to X$ to imply that X is isomorphic to Y.

Construction 2.6. Suppose that objects of $\mathscr{C}$ are non-empty sets and the morphisms are epimorphisms. Then, we define $\overline{\mathscr{C}}$ to be a category with the same objects as $\mathscr{C}$ but with extra morphisms $f : X \to Y$ whenever we can find a commutative diagram like the following one.

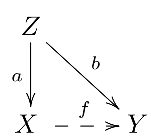


FIGURE 1. A diagram becomes commutative after adding $f : X \to Y$ such that $fa = b$, where a, b are morphisms in $\mathscr{C}$.

Remark 2.7. In practice, the category we are interested in usually does not have countably many equivalence classes. However, this construction may fill out the category to be a larger category with countably many equivalence by formally adjoining new morphisms. In $\overline{\mathscr{C}}$, morphisms are still epimorphisms.

When proving Adams' variant of Brown representability, we rely on a specific general inverse limit system to construct the representative for a given functor. However, we have to make sure that the general inverse limit is nonempty so that the functor is representable.

Proposition 2.8. *Suppose that the objects of* $\overline{\mathscr{C}}$ *(as in Construction 2.6) fall into countably many equivalence classes. Then,* $\varprojlim \mathscr{C}$ *is non-empty.*

We omit the proof since it is rather technical. For readers who wants to see the details, please check section two in Adams' original paper [1].

2.2. Properties of $\widehat{H}$.

We assume that H, as in 2.1, satisfies the Wedge and Mayer-Vietoris Axioms. We wish to prove that the extension of H, namely $\widehat{H}$, also satisfies the Wedge axiom and the Mayer-Vietoris Axiom to some extent.

Proposition 2.9. *$\widehat{H}$ satisfies the Wedge Axiom.*

Proof. Notice that inverse limits commute with coproduct. Then, because H satisfies the Wedge Axiom, $\widehat{H}$ also satisfies the Wedge Axiom. □

Lemma 2.10. *Let X be a CW-complex and $\{X_\alpha\}$ be a directed set of subcomplexes of X such that their union is X. Then, the canonical map*

$$\widehat{H}(X) \to \varprojlim_{\alpha} \widehat{H}(X_\alpha)$$

is an isomorphism.

We omit the proof since it is easy. For non-experienced readers, please read Corollary 23.13 in Miller's "Lectures on Algebraic Topology" [7].

We develop the following lemma in order to prove the Mayer-Vietoris Axiom on $\widehat{H}$ in a specific case.

Lemma 2.11. *Assume that K is a finite complex, and L, M are subcomplexes. Then, the following sequence is exact and natural on K, L, M:*

$$H(L) \times H(M) \xleftarrow{(i_1^*, i_2^*)} H(L \cup M) \longleftarrow H(S(L \cap M)) \xleftarrow{g^*} H(S(L \vee M)),$$

where the homomorphism g^* is induced by a map $g : S(L \cap M) \to S(L \vee M)$.

We omit the proof here and refer it to p.191 in Adam's paper [1].

The following proposition is essential for the entire proof of Theorem 2.1 and 2.3. It is the reason that the extended functor $\widehat{H}$ has a compatible $u \in \widehat{H}(Y)$ with Y satisfying some desirable qualities. It is also the place in the proof that uses the assumption that H takes values in the category of groups.

Proposition 2.12. *Let U, V be CW-complexes. If $U \cap V$ is a finite sub-complex of $U \cup V$, then the following diagram*

$$\begin{array}{ccc} H(U \cap V) = \widehat{H}(U \cap V) & \longleftarrow & H(U) \\ \uparrow & & \uparrow \\ \widehat{H}(V) & \longleftarrow & \widehat{H}(U \cup V) \end{array}$$

satisfies the Mayer Vietoris Axiom in the sense of Axiom 1.5.

As an immediate corollary, we restate the above proposition in terms of the exactness of cofibrations of $\widehat{H}$.

Corollary 2.13. *For any cofibering of CW-complexes*

$$K \xrightarrow{f} L \xrightarrow{i} M_f = L \cup_f CK,$$

such that K is a finite, the following sequence is exact after applying $\widehat{H}$:

$$H(K) = \widehat{H}(K) \xleftarrow{f^*} \widehat{H}(L) \xleftarrow{i^*} \widehat{H}(M_f).$$

Proof of Proposition 2.12. Let U_α, V_β run through all finite subcomplexes of U, V containing $U \cap V$. The assumption says that there exist $u \in \widehat{H}(U)$ and $v \in \widehat{H}(V)$ such that they restrict to the same element in $\widehat{H}(U \cap V)$; in the language of U_α, V_β, it is equivalent that u_α, v_α restrict to the same element in $H(U \cap V)$ for every α and β in a compatible fashion (respecting inclusion maps between $V_\alpha, V_{\alpha'}$ and $U_\beta, U_{\beta'}$).

For a given pair of α and β, we denote $H_{\alpha,\beta}$ as the set of elements in $H(U_\alpha \cup V_\beta)$ such that they restrict to u_α and v_β under the induced inclusions. Then, let $\mathscr{C}$ be the category with $H_{\alpha,\beta}$ as objects and $f : H_{\alpha',\beta'} \to H_{\alpha,\beta}$ induced by the inclusion $(\alpha \cup \beta) \subset (\alpha' \cup \beta')$ as morphisms. One can quickly check that $\mathscr{C}$ satisfies the two conditions in Section 2.1. Then, it suffices to prove that $\varprojlim \mathscr{C}$ (as defined in 2.4) is nonempty. But we have already established Proposition 2.8, so that we only need to show that the morphisms in $\mathscr{C}$ are epimorphisms and objects in $\overline{\mathscr{C}}$ fall into countably many equivalence classes.

In Lemma 2.11, we take $L = U_\alpha, M = V_\beta, K = L \cup M$ to get an exact sequence

$$H(U_\alpha) \times H(V_\beta) \xleftarrow{(i_1^*, i_2^*)} H_{\alpha,\beta} \longleftarrow H(S(U \cap V)) \xleftarrow{g^*} H(S(U_\alpha \vee V_\beta)),$$

where $U \cap V = U_\alpha \cap V_\beta$ by construction. From the middle homomorphism, we note that there is a homomorphism from $H(S(U \cap V))$ to $H_{\alpha,\beta}$, where we start to use the assumption that H takes values in the category of groups. Furthermore, the homomorphism is surjective through the definition of $H_{\alpha,\beta}$ and the exactness of the sequence. From the naturality of the exact sequence, the homomorphism commutes with the map $i^* : H_{\alpha',\beta'} \to H_{\alpha,\beta}$ as shown in the following diagram. From Figure 2, we note that i^* is an epimorphism.

$$\begin{array}{ccc} H(S(U \cap V)) & \xrightarrow{a} & H_{\alpha,\beta} \\ \uparrow \approx & & \uparrow i^* \\ H(S(U \cap V)) & \xrightarrow{b} & H_{\alpha',\beta'} \end{array}$$

FIGURE 2. Commutativity of the homomorphism and morphisms in $\mathscr{C}$. Our assumption implies $H(S(U \cap V)) \approx H(S(U_\gamma \cap V_\delta))$. Then, since a is surjective, i^* is surjective.

Next, we consider $\overline{\mathscr{C}}$ and state the following property, which can be proved with the provided group structure and the right side of the exact sequence.

Lemma 2.14. *There is a morphism $H_{\alpha',\beta'} \to H_{\alpha,\beta}$ in $\overline{\mathscr{C}}$ if and only if the image of $g^*_{\alpha',\beta'}$ is contained in the image of $g^*_{\alpha,\beta}$, where g^* is mentioned in Lemma 2.11. Moreover, $H_{\alpha',\beta'}$ and $H_{\alpha,\beta}$ are equivalent if and only if $g^*_{\alpha',\beta'}$ equals to $g^*_{\alpha,\beta}$.*

Now, we can use the number of different images of g^* to count the number of equivalence classes in $\overline{\mathscr{C}}$. Instead of running over all $H_{\alpha,\beta}$, we make an overestimate. We consider the countable set of all finite simplicial CW-complexes $\{K_\alpha\}$ such that every finite CW-complex is homotopic equivalent to one of those K_α. Then, for each K_α, the maps $g : S(U \cap V) \to K_\alpha$ fall into countable many homotopy classes by the simplicial approximation theorem (we approximate $S(U \cap V)$ by an equivalent finite simplicial complex).

Therefore we see that in total there are still only a countably many images of homomorphisms

$$g*: H(K) \to H(S(U \cap V)),$$

so that objects of $\mathscr{C}$ fall into a countable number of equivalence classes.

Hence, by Proposition 2.8, $\varprojlim \mathscr{C}$ is nonempty, which completes the proof. □

2.3. **Representability in *CW*.** First, we mention a lemma of Adams without the proof (please refer to p.194 in [1]).

Lemma 2.15. *The following two 1-1 correspondences are natural for maps of Y:*

$$\begin{aligned}\widehat{H}(Y) \approx &Nat\ Trans([X,Y]_a, \widehat{H}(X)) \\ \approx &Nat\ Trans([K,Y], H(K)),\end{aligned}$$

where X is any CW-complex and K is any finite CW-complex.

The above lemma has the form of Yoneda correspondence. However, to see an honest proof, one has to study how inverse limit interacts with "almost homotopy" (defined in 2.2). We refer readers to Adams' paper for those information. Notationally, we call elements of Nat Trans$([X,Y]_a, \widehat{H}(X))$ as T_u and elements of Nat Trans$([K,Y], H(K))$ as $\widehat{T}_u$ with $u \in \widehat{H}(Y)$.

We start by proving Theorem 2.1 through three steps.

Proof of Theorem 2.1. From the second correspondence in Lemma 2.15, it suffices to construct a proper pair (Y,u) so that T_u is a natural isomorphism. Here, we prove the following more general version of Theorem 2.1: let Y_0 be any CW-complex and let $u_0 \in \widehat{H}(Y_0)$. Then, there exist (Y,u) and an embedding $i: Y_0 \to Y$ such that u restricts to u_0 along the map induced by i, and T_u is a natural isomorphism of sets restricted to $S^n \in \mathcal{S}$. In this case, by taking Y_0 to be a point, the result obviously implies Theorem 2.1.

Step I. Surjectivity of T_{u_1}. To start, we choose Y_0 to be an arbitrary CW-complex (not necessarily finite) with an arbitrary element $u_0 \in \widehat{H}(Y_0)$. We will try to construct $Y_0 \subset Y_1 \subset Y_2 \subset Y_3 \subset ...$ with corresponding $u_i \in \widehat{H}(Y_i)$ so that $Y = \bigcup Y_i$ together with some u will give the desired natural isomorphism.

We let S run through the set of all n-spheres $\mathcal{S}$, and h run through $\widehat{H}(S) = H(S)$ for each S. Define Y_1 as

$$Y_1 := Y_0 \vee \bigvee_{S,h} S.$$

Then, the choice of u_1 is given by Proposition 2.9 so that $u_1 \in \widehat{H}(Y_1)$ restricts to u_0 and h through the obvious inclusions.

At this stage, we notice that the natural transformation

$$T_{u_1} : [S, Y_1] \to H(S)$$

is already an epimorphism of sets for spheres $S \in \mathcal{S}$.

Step II. Injectivity of T_{u_n}. Given Y_n a CW-complex with an element $u_n \in \widehat{H}(Y_n)$, we want to construct (Y_{n+1}, u_{n+1}) with the embedding $i: Y_n \to Y_{n+1}$ satisfying that u_{n+1} restricts to u_n along i^*. Moreover, we want that if $f, g: S \to Y_n$ are two maps from a sphere S to Y_n that represent the same element, namely $f^*(u_n) = g^*(u_n)$, then they are reduced in Y_{n+1}, namely $if \approx ig$ in Y_{n+1}.

For each pair of homotopy classes of maps $f, g : S^n \to Y_n$ such that $f^*(u_n) = g^*(u_n)$, we take a tuple (f, g). As S^n runs through $\mathcal{S}$, we construct the index set of all those tuples, which we call A. Now, for each $\alpha \in A$, we have a pair of maps $f_\alpha, g_\alpha : S_\alpha \to Y_n$ that represents some same element in $H(S_\alpha)$.

We let α run through A to define

$$Y_{n+1} = Y_n \cup_{f_\alpha, g_\alpha} \bigcup_{\alpha \in A} (I \times S_\alpha / I \times pt) \approx M_h,$$

where we attach the reduced cylinder to Y_n along f_α on the one end and g_α on the other end. Also, the second equality says that Y_{n+1} is homotopic equivalent to the mapping cone of $h := \bigvee_{\alpha \in A}(f_\alpha \vee g_\alpha)$. Thus, we intentionally guarantee that if_α is homotopic equivalent to ig_α in Y_{n+1}.

It remains to find an appropriate $u_{n+1} \in \widehat{H}(Y_{n+1})$ such that u_{n+1} restricts to u_n. At first, it is tempting to directly use the exactness of $\widehat{H}$ on the cofibration sequence

$$R := \bigvee_{S \in \mathcal{S}} (S \vee S) \xrightarrow{f} Y_{n+1} \xrightarrow{i} M_h.$$

However, recall from Corollary 2.13 that the exactness is only preserved when R is a finite CW-complex. We use Zorn's lemma to resolve this technical difficulty.

For the setup, we consider the pair (B, h) where $B \subset A$ and h is taken from

$$\widehat{H}(Y_n \cup_{f_\beta, g_\beta} \bigcup_{\beta \in B} (I \times S_\beta / I \times pt)),$$

such that h restricts to u_n. Notice that there is a partial ordering; i.e., $(B, h) \leq (B', h')$ if $B \subset B'$ and h' restricts to h.

To apply Zorn's lemma for the existence of a maximal element, we need to check two things. First, the set of $\{(B, h)\}$ is nonempty because we may take $B = \emptyset$ and $h = u_n$. Second, we need to show that every chain has an upper bound. Indeed, for any chain $\{(B_\alpha, h_\alpha)\}_{\alpha \in C}$, we take $B = \bigcup B_\alpha$ and there exists $h \in \widehat{H}(B)$ such that it restricts to h_α for each $\alpha \in C$ by the isomorphism established in Lemma 2.10.

Then, by Zorn's lemma, we know the existence of a maximal element (M, m) with the associated mapping cone M_m. To successfully find $u_{n+1} \in \widehat{H}(Y_{n+1})$, it suffices to prove that $M = A$. For any $\alpha \in M \neq A$, we have an "extended" cofibration sequence

$$S_\alpha \vee S_\alpha \xrightarrow{f} M_m \xrightarrow{i} M'_m,$$

where $M'_m \approx M_m \cup_{f_\alpha, g_\alpha} (I \times S_\alpha / I \times pt)$.

Now, we may use Corollary 2.13 to find u_{n+1} that restricts to u_n.

Because we assume (M, m) to be maximal, we deduce $M = A$ and take $u_{n+1} = m$. Inductively, we construct (Y_n, u_n) for all $n \geq 2$, given the base case (Y_1, u_1).

Step III. Bijectivity of T_u. We set

$$Y := \bigcup Y_n$$

and let $u \in \widehat{H}(Y)$ be the element that restricts to u_n for each n (the existence is given by Lemma 2.10). We already know that the corresponding natural transformation T_u is an epimorphism of sets (guaranteed at (Y_1, u_1)). Also, if $f, g : S \to Y$ are two maps that represents the same element in $H(S)$ meaning $f^*(u) = g^*(u)$, since S is finite, then f and g must map into some Y_n for some n so that $f^*(u_n) = g^*(u_n)$. By construction, f is homotopy equivalent to g in Y_{n+1}. Thus, we conclude that f is homotopy equivalent to g in Y, which implies that T_u is a natural isomorphism in the context of sets. □

We proceed to prove Theorem 2.3.

Proof of Theorem 2.3. **Step IV**. **Extension**. (i) We would first like to extend the natural isomorphism on $\mathcal{S}$ to a natural isomorphism on CW. Note that as in the proof of Proposition 2.12, it suffices to consider finite CW-complexes K of different homotopy types. Since $[S, -]$ determines those K, one can show that

$$T_u : [K, Y] \approx H(K)$$

is a natural isomorphism of sets, where K is any finite CW-complex.

From Lemma 2.15, with a pair (Y, u) so that T_u is a natural isomorphism of sets, there is a unique natural transformation of sets

$$\widehat{T}_u : [X, Y]_a \to \widehat{H}(X),$$

which extends T_u meaning $\widehat{T}_u = T_u$ when $X \in \widetilde{CW}$ is finite.

(ii) We claim that the natural transformation $\widehat{T}_u$ is a natural isomorphism of sets. To prove the claim, we first prove $\widehat{T}_u$ is monomorphic.

We let $f, g : X \to Y$ be two maps such that $\widehat{T}_u(f) = \widehat{T}_u(g)$ (they represent some same element of $\widehat{H}(X)$). Then, we choose any finite CW-complex K and map $h : K \to X$. By Lemma 2.15, we have

$$h^*\widehat{T}_u(f) = h^*\widehat{T}_u(g),$$

which can also be written as $T_u(hf) = T_u(hg)$. Now, we are ready to use the result proved in Theorem 2.1, namely that T is a natural isomorphism. It immediately follows that $hf \approx hg$ for an arbitrary $h : K \to X$. By the definition of almost homotopy, we conclude that $f \approx_a g$ as desired.

To prove that $\widehat{T}$ is epimorphic, we invoke the stronger statement used in the proof of Theorem 2.1, which gives the freedom of choosing an arbitrary initial CW-complex Y_0. Let X be any CW-complex and $x \in \widehat{H}(X)$. Then, we let $Y_0 = X \vee Y$ and $u_0 \in \widehat{H}(Y_0)$ be the element that restricts to u in $\widehat{H}(Y)$ and x in $\widehat{H}(X)$ (the choice is given by the Wedge Axiom on $\widehat{H}$). Using the result we proved, we have (Y', u') and $i : Y_0 \to Y'$ such that $T'_{u'}$ is a natural isomorphism of sets, and u' restricts to u_0 along i^*. Specifically, we have the commutativity of the following diagram

$$\begin{array}{ccc} [S^n, Y] & & \\ \downarrow{\scriptstyle i^*} & \searrow^{T_u} & \\ [S^n, Y'] & \xrightarrow{T_{u'}} & H(S^n). \end{array}$$

Here, since T_u and $T_{u'}$ are isomorphism of sets, then i^* is an isomorphism of homotopy groups. We let S^n runs through $\mathcal{S}$ to note that there is a weak homotopy equivalence between Y and Y'. By Whitehead's Theorem, Y and Y' are homotopy equivalent so that we have an inverse map $i^{-1} : Y' \to Y$.

We compose the injection $X \to Y'$ and $i^{-1} : Y' \to Y$ to obtain a map f from X to Y so that $f^*(u) = x$ by construction. Therefore, we conclude that $\widehat{T}$ is epimorphic, which completes the proof of (ii). □

2.4. **Spectrum-Cohomology Correspondence.** We assume readers' familiarity with the definition of general homology theories and cohomology theories (see Concise [6] for details). As a quick remainder, to define a general homology theory H_* or a general cohomology theory H^* in the category of topological spaces, it suffices to define them on the category of CW-complexes, namely CW. Besides, the differences between reduced and unreduced (co)homology theories is not significant in our discussion, since they are equivalent up to adjoining base-point.

We state the definition of an $\Omega - specturm$.

Definition 2.16. An Ω-*spectrum* is a sequence of based space T_n and weak homotopy equivalences $\sigma : T_n \to \Omega T_{n+1}$.

Theorem 2.17. *Let H^* be a general cohomology theory defined on the category of topological spaces (without any assumption on the cardinality of the coefficient groups). Then, H^* is the general cohomology theory corresponding the an Ω-spectrum T.*

Remark 2.18. Note that we have already done most of the work in Theorem 2.3 on the level of each cohomology functor H^n in the category of connected CW-complexes.

Proof. We take usage of the suspension axiom in the definition of a general cohomology theory to tie together the representability of H^n on different dimensions.

Here, we assume the reduced version of cohomology. Notice that suspension in CW is always connected and we have natural isomorphisms of suspension in the axiom, it suffices to restrict our attention to connected CW-complexes.

A general cohomology theory on each dimension assumes the Wedge Axiom. Further, H^n satisfies the Mayers-Vietories Axiom as a property deduced from other axioms. Therefore, we invoke Theorem 2.3 so that there is a unique natural isomorphism

$$T : [-, Y_n] \approx H^n(-)$$

up to weak homotopy equivalence. It remains to show that the natural isomorphism $H^n(X) \approx H^{n+1}(\Sigma X)$ corresponds to a weak homotopy equivalence $\sigma_n : Y_n \to \Omega T_{n+1}$.

First, by the naturality of T, the natural isomorphism $H^n(X) \approx H^{n+1}(\Sigma X)$ corresponds to $[X, Y_n] \approx [\Sigma X, Y_{n+1}] = [X, \Omega Y_{n+1}]$, where the second equality is by the topological adjointness. Specifically, we denote the composed natural isomorphism $[X, Y_n] \approx [X, \Omega Y_{n+1}]$ as $\Theta(X)$.

To prove the theorem, it suffices to show $\sigma_n = \Theta(\mathbb{1}) : Y_n \to \Omega Y_{n+1}$ is a weak homotopy equivalence. For an arbitrary map $f : X \to Y_n$, we have the following commutative diagram by the natruality of Θ:

$$\begin{CD}
[Y_n, Y_n] @>{f^*}>> [X, Y_n] \\
@V{\Theta}VV @VV{\Theta}V \\
[Y_n, \Omega Y_{n+1}] @>{f^*}>> [X, \Omega Y_{n+1}]
\end{CD}$$

We notice the following equality from the above diagram

$$\Theta(f) = \Theta f^*(\mathbb{1}) = f^*\Theta(\mathbb{1}) = f^*(\sigma_n).$$

We lastly identify $f^*(\sigma_n)$ with $\sigma_n f$ on the CW-level. Therefore, by taking X to be n-spheres $S^n \in \mathcal{S}$, since Θ is a natural isomorphism, σ_n induces an isomorphism on $\pi_i(Y_n)$ and $\pi_i(\Omega Y_{n+1})$ for all i. Hence, σ_n is a weak homotopy equivalence. □

General homology theories are also representable on the spectrum-level. S-duality gives us the tool to convert the unfamiliar homology problem to the already solved cohomology representability. However, this conversion only gives us functors defined on finite CW-complexes. Therefore, one has to use the full power of Theorem 2.3 to deduce the homology representability.

We define a prespectrum that generates general homology theories.

Definition 2.19. An *prespectrum* is a sequence of based space E_n and weak homotopy equivalences $\sigma : \Sigma E_n \to E_{n+1}$.

Theorem 2.20. *Let H_* be a general homology theory defined on the category of topological spaces (without any assumption on the cardinality of the coefficient groups). Then, H_* is the general cohomology theory corresponding a prespectrum E.*

The above Theorem follows from Adams' theorems and G. Whitehead's proof of a weaker theorem in [10].

3. Neeman's Representability in Triangulated Categories

Indeed, Brown's and Adams' proof of representability is rather more categorical than topological. The same proof technique can applied to triangulated categories. In this new context, the wedge axiom is stated as sending small coproducts to products; and the Mayer-Vietoris Axiom means that the functor is "homological". By restricting our attention to compactly generated triangulated categories, we can obtain the parallel result of representability by Neeman in [8]. The preparation and the proof are presented in a similar four-steps structure to highlight the common theme of proving representability. Readers are particularly encouraged to compare two proofs line-to-line for more insight.

3.1. **Compactly generated Triangulated Category.**

In the following definition, $\mathcal{T}$ is referred to be a triangulated category. For readers who do not know triangulated categories, please see the chapter on triangulated categories in the book Categories and Sheaves [5].

Definition 3.1. A triangulated category $\mathcal{T}$ is said to *contain small coproducts* if for any small set Λ, any collection $\{t_\lambda \in \mathrm{Ob}(\mathcal{T}) \mid \lambda \in \Lambda\}$ has a coproduct

$$\coprod_{\lambda \in \Lambda} t_\lambda.$$

Definition 3.2. An object c of $\mathcal{T}$ is called *compact* if for any coproduct in $\mathcal{T}$, the following equality holds:

$$\mathrm{Hom}_{\mathcal{T}}(c, \coprod_{\lambda \in \Lambda} t_\lambda) = \coprod_{\lambda \in \Lambda} \mathrm{Hom}_{\mathcal{T}}(c, t_\lambda).$$

Definition 3.3. A triangulated category is called *compactly generated* if $\mathcal{T}$ contains small coproducts and there exists a small set T of compact objects such that

$$\mathrm{Hom}(t, x) = 0 \text{ for all } t \in T \implies x = 0.$$

Furthermore, we say that T is a *generating set* if T is also assumed to be closed under suspension.

We need T to detect the zero element and generates $\mathcal{T}$; as an analogy, T plays the same role as a basis of a vector space and the set of n-spheres in CW.

3.2. **Representability in $\mathcal{T}$.** We are ready to state the representability theorem in the context of triangulated categories. Note that this version is analogous to (i) in Brown's Theorem 1.11, in which the functor is defined on the entire category rather than a full, well-behaved subcategory.

As for the sufficient conditions, we change the Mayers-Vietoris Axiom to be homological and generalize the Wedge Axiom to sending small coproducts to products.

Theorem 3.4 (Neeman [8]). *Let $\mathcal{T}$ be a compactly generated triangulated category with the generating set T. Let $H : \mathcal{T}^{op} \to Ab$ be a homological functor (taking triangles to long exact sequences). Also, we require the following natural isomorphism*

$$H(\coprod_{\lambda\in\Lambda} t_\lambda) \approx \prod_{\lambda\in\Lambda} H(t_\lambda)$$

for small coproducts in $\mathcal{T}$. Then, H is representable in the sense of Definition 1.1.

Remark 3.5. Putting Construction 1.8 into the new context, for fixed $R \in \mathcal{T}$ and $\alpha \in H(R)$, we have the following transformation $P_\alpha : [-, R] \to H(R)$ defined by

$$P_\alpha(f) = H(f)\alpha,$$

where $f \in [X, R]$.

Moreover, by Yoneda's Lemma, we have a one to one correspondence:

$$H(R) \approx \text{ Nat Trans}([-, R], H(-)).$$

Thus, it suffices to find appropriate R and α so that P_α is a natural isomorphism.

In the proof of Theorem 3.4, we aim to draw direct comparisons between representability in CW-complexes and representability in triangulated categories. Recall that before proving Theorem 2.3, we first proved Theorem 2.1 on finite CW-complexes (containing n-spheres) since CW-complexes in general are hard to control. Likewise here, we first prove the following theorem.

Theorem 3.6 (Neeman [8]). *Let $\mathcal{T}$ and H satisfy the same assumptions as in the statement of Theorem 3.4. Then, for all $t \in T$, there exists*

$$P : [t, R] \approx H(t),$$

which is an isomorphism of sets.

Proof of Theorem 3.6. The overall strategy is similar to the proof of Representability theorem in the topological context. To construct a desirable R, we first construct a sequence of tuples (R_n, α_n), where $\alpha_n \in H(R_n)$. Then, we take the homotopy colimit of R_n to be our R and choose an appropriate α.

Step I. Surjectivity of P_{α_1}. We formally define the set of values of H on T as

$$U_1 = \bigcup_{t\in T} H(t).$$

Notice that we can think of element of U_1 as (α, t) such that $\alpha \in H(t)$, where t is attached to keep track of where α comes from.

Then, we can define

$$R_1 = \coprod_{(\alpha,t)\in U_1} t.$$

Since we assume that H sends small coproducts to products, we have

$$H(R_1) \approx \prod_{(\alpha,t)\in U_1} H(t).$$

In other words, for an arbitrary element $\alpha \in H(t)$ (subsequently $(\alpha, t) \in U_1$), we have an unique element $\alpha_1 \in H(R_1)$ satisfying the following property: if $t \to R_1$ maps t to $t \in R_1$ that corresponds to $(\alpha, t) \in U_1$, then the induced map $H(R_1) \to H(t)$ restricts $\alpha_1 \in H(R_1)$ to $\alpha \in H(t)$.

Then, by Yoneda's Lemma, the element α_1 gives rise to a natural transformation

$$P_{\alpha_1} : [-, R_1] \to H(-).$$

And the discussion we had about α_1 says precisely that

$$P_{\alpha_1}(t) : [t, R_1] \to H(t)$$

is epimorphic for all $t \in T$.

Step II. Injectivity of P_{α_n}. Given $R_n \in \mathcal{T}$ with an element $\alpha_n \in H(R_n)$, we want to construct (R_{n+1}, α_{n+1}) with a map k from R_n to R_{n+1} such that α_{n+1} restricts to α_n along i^* and the following condition is satisfied. For any two maps $f, g \in [t, R_n]$ that represent the same element in the sense that $f^*(\alpha_n) = g^*(\alpha_n)$, the extended maps $kf = kg$ are the same.

To better utilize the properties of triangulated category, we rephrase the above property in terms of kernels. Specifically, we define

$$U_{n+1} = \bigcup_{t\in T} \ker\{P_{\alpha_n}(t) : [t, R_n] \to H(t)\}.$$

We desire that if a map $f : t \to R_n$ is in U_{n+1}, then $kf = 0$.

Again, we can think of elements of U_n as (f, t), where $f \in \ker\{P_{\alpha_n}(t)\}$. Then, we form a corresponding "kernel set" in $\mathcal{T}$:

$$K_{n+1} = \coprod_{(f,t)\in K_{n+1}} t,$$

and let $\coprod_{(f,t)\in K_{n+1}} f : K_{n+1} \to R_n$ be the coproduct of all maps f for $(f, t) \in K_{n+1}$.

Then, analogous to how we use the cofibration sequences and mapping cones to define Y_{n+1}, here we define R_{n+1} using the first property of an exact triangle. Let R_{n+1} be given by the following triangle:

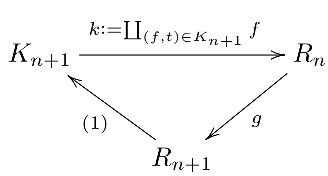


FIGURE 3. The exact triangle defining R_{n+1}

As for constructing α_{n+1}, we note the isomorphism guaranteed by assumption

$$H(R_n) \to H(K_{n+1}) = H\left(\coprod_{(f,t)\in K_{n+1}} t\right) \approx \prod_{(f,t)\in K_{n+1}} H(t).$$

Then, α_n maps to zero as we chose f to be in the kernel of $P_{\alpha_n}(t)$.

Because H is homological, we obtain an exact sequence following from the exact triangle

$$H(R_{n+1}) \xrightarrow{i} H(R_n) \xrightarrow{j} H(K_{n+1}),$$

which should remind readers of the cofibration sequence in the topological case. Directly from the exactness, since $j(\alpha_n) = 0$, there exists $\alpha_{n+1} \in H(R_{n+1})$ such that $i(\alpha_{n+1}) = \alpha_n$.

To see that such construction of (R_{n+1}, α_{n+1}) kills off the redundancy of representation on the R_n level, for an arbitrary $t \in T$, we take $f \in \ker\{P_{\alpha_n}(t)\}$. By construction, f factors through h in the triangle in Figure 3; i.e., $kf = 0$.

So far, we have already given the construction of (R_n, α_n) defined inductively for $n \geq 2$ with (R_1, α_1) well-defined.

Step III. Bijectivity of P_α. We set

$$R = \underrightarrow{\mathrm{hocolim}}\, R_n,$$

where R is the homotopy colimit of R_n. We proceed to find appropriate $\alpha \in H(R)$ such that α restricts to α_n for all n.

Indeed, we consider the triangle

$$\begin{array}{ccc} \coprod_n R_n & \xrightarrow{\text{1-shift}} & \coprod_n R_n \\ & {\scriptstyle (1)}\nwarrow \quad \swarrow & \\ & R = \underrightarrow{\mathrm{hocolim}}\, R_n & \end{array} \quad .$$

We apply H to the above triangle to obtain an exact sequence,

$$\begin{array}{ccccc} H(R) & \longrightarrow & H(\coprod_n R_n) & \xrightarrow{\text{1-shift}} & H(\coprod_n R_n)\ , \\ & & \cong \uparrow & & \cong \uparrow \\ & & \coprod_n H(R_n) & \xrightarrow{\text{1-shift}} & \coprod_n H(R_n) \end{array}$$

where we use the isomorphism of products to simplify it. Formally, $\coprod_n \alpha_n$ is in the kernel of the 1-shift so that there exists $\alpha \in H(R)$ mapping to it. Thus, α restricts to α_n by an induced map of the inclusion of R_n into R. Rephrasing the above discussion through Yoneda correspondence, we have the commutative diagram

Finally, with the construction of (R, α), we wish to show that $P_\alpha(t)$ is an isomorphism for all $t \in T$. First, we note that on level R_1, we already obtained surjectivity. Formally, we substitute R_n in Figure 4 by R_1 and take values of an arbitrary $t \in T$. Since we know $[t, R_1] \to H(t)$ is surjective from Step I., the map $[t, R] \to H(t)$ must also be surjective.

For injectivity, like in Step III of proving Adams' theorems, we need an "approximation lemma" to relate R_α to R_{α_n}. Let $f \in [t, R]$ such that $P_\alpha(t)(f) = 0$. We cite the "approximation lemma" from [8] without the proof.

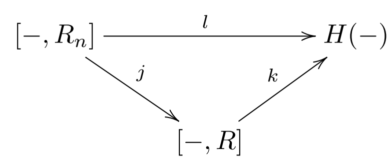


FIGURE 4. The commutative diagram about $[-, R]$ and $[-, R_n]$, where l corresponds to α_n, k corresponds to α, and j corresponds to the restriction of α to α_n in Yoneda's lemma.

Lemma 3.7. *Suppose c is a compact object of $\mathcal{T}$, and we have a sequence of objects and morphisms in $\mathcal{T}$:*

$$X_1 \to X_2 \to X_3 \to \ldots$$

We assume that $\mathcal{T}$ admits small coproducts. Then, we have the equality

$$[c, \underrightarrow{\text{hocolim}}\, X_i] = \underrightarrow{\text{colim}}[c, X_i].$$

Back to the proof of injectivity, from the lemma, we have

$$[c, R] = \underrightarrow{\text{colim}}[c, R_n].$$

In other words, there exists $f_n : t \to R_n$ that approximates f in the sense that the composite

$$t \xrightarrow{f_i} R_n \longrightarrow R$$

is f. Again, we examine Figure 4 and note that $j(f_n) = f$ while $k(f) = 0$. Thus, f_i is forced to be in the kernel of $P_{\alpha_n}(t)$, which is $(f_n, t) \in U_{n+1}$.

From Step II., we recall that $gf_n = 0$ in the sense of the triangle in Figure 3, but the composition

$$R_n \xrightarrow{g} R_{n+1} \xrightarrow{g^*} R, \text{ such that } f = (g^* g) f_n = g^*(g f_n) = 0.$$

Hence, we complete the proof of Theorem 3.6. □

We are ready to prove Theorem 3.4 now.

Proof of Theorem 3.4. With the representability on the generating set T, to prove Theorem 3.4, we extend the representability to $\mathcal{T}$.

Step IV. Extension Let $\mathcal{U}$ be the minimal full subcategory of objects $X \in \mathcal{T}$ such that for all $n \in \mathbb{Z}$, the map $T_\alpha(\sum^n X) : [\sum^n X, R] \to H(\sum^n X)$ is an isomorphism. Then, obviously, we note that from Theorem 3.6, $\mathcal{U}$ contains the compact generating set T. Besides, $\mathcal{U}$ is closed with respect to $\mathcal{T}$-coproducts of its objects directly from the definition. Note that to finish the proof, it suffices to show $\mathcal{U} = \mathcal{T}$.

Let Z be an arbitrary object of $\mathcal{T}$. We try to show $Z \in \mathcal{U}$. To start, note that

$$H := [-, Z]_{\mathcal{T}}$$

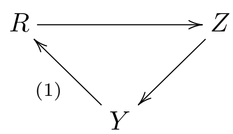
is a homological functor on $\mathcal{U}$. Since from our assumption, $\mathcal{U}$ is compactly generated with the generating set T, we may apply the representability restricted to T so that we have a pair $(R, \alpha), R \in \mathcal{U}, \alpha \in [R, Z]_{\mathcal{T}}$ and a natural transformation

$$T_\alpha : [-, R]_{\mathcal{U}} \to [-, Z]_{\mathcal{T}},$$

which is an isomorphism when the first component is $t \in T$. Moreover, this natural transformation must be restricted to a natural isomorphism on a full, triangulated further subcategory of $\mathcal{U}$ containing T and closed with respect to $\mathcal{U}$-coproducts.

However, we already assumed that $\mathcal{U}$ is minimal with this condition. Thus, T_α is a natural transformation on $\mathcal{U}$. Now, we use Yoneda's Lemma in $\mathcal{T}$ to note that the above natural isomorphism must correspond to a morphism $R \to Z$. We will use this special morphism to complete the rest of the proof.

We complete $R \to Z$ to a triangle in $\mathcal{T}$ in the following sense.



It follows directly from the property of exact triangles that for $s \in \mathcal{U}$, $[s, Y] = 0$. Specifically, since $T \subset \mathcal{U}$, $[t, Y] = 0$. By the definition of T, we deduce that $Y = 0$, which implies that $R \to Z$ is an isomorphism. Finally, since $\mathcal{U}$ is full and R is an object of $\mathcal{U}$, Z must be an object of $\mathcal{U}$.

In conclusion, we showed that the natural transformation in Theorem 3.6 is a natural isomorphism to complete the proof. □

Remark 3.8. The Spectrum-(co)homology correspondence Theorems 2.17 and 2.20 are implied by Theorem 3.4 since the stable homotopy category is known to be a compactly generated triangulated category. Thus, the triangulated category representability generalizes the classic topological representability theorem.

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