WHAT GROUPS ARE AMENABLE, AND WHAT ARE NOT

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Abstract. We define what does it mean for a group to be amenable, and discuss examples of amenable groups and non-amenable groups. Moreover, we prove the Inheritance Theorem of Amenability to decide the amenability of more groups we have studied in class.

Contents

1. What does it mean for a group to be Amenable

First, for X a set, we denote $\mathcal{L}^\infty(X,\mathbb{R})$ as the set of all bounded functions $f: X \to \mathbb{R}$. Note that $\mathcal{L}^\infty(X,\mathbb{R})$ is a $\mathbb{R}-$vector space with point-wise addition and scalar multiplication.

Another remark is that a left $G-$action on X induces a left $G-$action on $\mathcal{L}^\infty(X,\mathbb{R})$ in the following way:

$$\varphi : G \times \mathcal{L}^\infty(X,\mathbb{R}) \to \mathcal{L}^\infty(X,\mathbb{R})$$
$$\varphi(g,f) : x \mapsto f(g\cdot x), \text{ for } x \in X.$$

Note that φ is indeed an action on $\mathcal{L}^\infty(X,\mathbb{R})$ since first, we note that the identity condition is satisfied by

$$e \cdot f(x) = f(e\cdot x) = f(x).$$

The transitivity condition is satisfied by

$$g\cdot(k\cdot f)(x) = g \cdot f(k\cdot x) = f(g\cdot(k\cdot x)) = f(gk\cdot x)) = gk\cdot f(x),$$

where $g,k \in G$.

An immediate example is that the left translation of G defined by $h \mapsto gh$ for fixed $g \in G$ and $\forall h \in G$ induced an action of G on $\mathcal{L}^\infty(G,\mathbb{R})$, which we call the *induced left translation on* $\mathcal{L}^\infty(G,\mathbb{R})$.

Definition 1.1 (Amenable Groups)**.** A group is *amenable* if there is a $G-$invariant mean on $\mathcal{L}^\infty(G,\mathbb{R})$, i.e., an $\mathbb{R}$-linear map $m : \mathcal{L}^\infty(G,\mathbb{R}) \to \mathbb{R}$ such that the following properties are satisfied:

(i) *(Normalization)* $m(f_1) = 1$, where f_1 is the constant 1-map,

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(ii) *(Positivity)* $m(f) \geq 0$ if $f \geq 0$ pointwise,
(iii) and *(Left-invariance)* $m(g \cdot f) = m(f)$ where $g \cdot f$ is the induced left translation on $\mathcal{L}^\infty(G, \mathbb{R})$

Remark 1.2. The linearity condition is defined based on the vector space structure of $\mathcal{L}^\infty(G, \mathbb{R})$ as remarked earlier. In defining G- invariant mean later on, I will skip checking the linearity since it is straightforward to check.

Remark 1.3. Intuitively, we define a $G-$invariant mean, m, on the set of bounded real functions defined on groups to generalize the usual notion of means of functions defined on sets. Therefore, we require m to be $\mathbb{R}$-linear so that the mean respects the vector space structure of $\mathcal{L}^\infty(G, \mathbb{R})$.

The normalization and positivity conditions are two conditions a mean function should satisfy. The left-invariance condition makes sure that the mean is $G-$invariant so that upon left actions on G, the mean is fixed. I understand this property as if we graph a function $f \in \mathcal{L}^\infty(G, \mathbb{R})$, then by changing $x-$axis through a left translation (a permutation on G), the mean should remain unchanged.

2. Examples of (non)amenable groups

Proposition 2.1. *Finite groups are amenable.*

Proof. G be a finite group, then we can define m such that for $f \in \mathcal{L}^\infty(G, \mathbb{R})$,

$$m(f) = \frac{1}{|G|} \sum_{g \in G} f(g).$$

Notice that the normalization condition is satisfied since $m(f_1) = \frac{|G|}{|G|} = 1$. Also, one can check the positivity condition quite easily. For the left-invariance condition, we note that since G is a finite group,

$$\sum_{g \in G} f(g) = \sum_{g \in G} f(hg),$$

by Cayley's Theorem.

(Indeed, m in this case is called the *averaging operator* since it sums over all values of f and is divided by the order of G.) □

However, when it comes to infinite groups, whether a group is amenable or not become extremely hard to answer since the sum may become infinite and it makes no sense to divide something by an infinite number. Fortunately, because abelian groups are particularly nice, they are amenable.

Theorem 2.2. *Abelian groups are amenable.*

We omit the proof since it involves some essential theorems in functional analysis. To see why functional analysis is crucial here, the function space $\mathcal{L}^\infty(G, \mathbb{R})$ with the supremum norm is a Banach space by checking the completeness. When G is infinite, $\mathcal{L}^\infty(G, \mathbb{R})$ is infinite dimensional so that results in functional analysis become essential. Specifically, the key tool in the proof uses *Markov-Kakutani fixed point theorem*, for which I have a simple proof only using Hahn-Banach. For folks who want to see details, please feel free to reach out to me or read p.291 in [1].

In the Merriam-Webster Dictionary, ”amenable” is defined as ”liable to be brought to account”. In the context of groups, amenability recognizes the ”finiteness” of the group structure; i.e., can subgroups entirely preserve the structure of its parent group? According to this intuition, we can conjecture free groups to be not amenable since free groups contain free groups of any index as subgroups.

Proposition 2.3. *The free group F_2 is not amenable.*

Proof. Towards a contradiction, assume that $F_2 = \{a, b \mid\}$ is amendable with a left invariant mean m. Let A be the set of freely reduced words that start with a non-trivial power of a. Then, we observe that

$$A \cup a^{-1} \cdot A = F_2.$$

Recall that the characteristic function of $S \subset X$, denoted by χ_S, sends S to 1 and $X - S$ to 0. In other words, the characteristic function indicates when elements are in the subset. Therefore, an immediate property of characteristic functions is $\chi_A + \chi_B = \chi_{A\cup B}$ if A and B are disjoint. Moreover, we claim that

$$m(\chi_A + \chi_B) \geq m(\chi_{A\cup B}).$$

Since the difference $(\chi_A + \chi_B) - \chi_{A\cup B}$ is non-negative, we have $m[(\chi_A + \chi_B) - \chi_{A\cup B}] \geq 0$ by the positivity property of a left invariant mean. Then, by the $\mathbb{R}$ linearity of m, we see the claim.

We apply χ to A, $a^{-1} \cdot A$, and F_2 to note that

$$\begin{aligned} 1 = m(1) = m(\chi_{F_2}) &\leq m(\chi_A + \chi_{a^{-1}\cdot A}) \\ &= m(\chi A) + m(\chi_{a^{-1}\cdot A}) \\ &= m(\chi A) + m(a^{-1} \cdot \chi_A) \\ &= 2m(\chi A) \\ \implies m(\chi A) \geq 1/2, \end{aligned}$$

where the second to last equality holds by writing out the definitions and the last equality holds by the left-invariance property of m.

On the other hand, we also have that A, $b \cdot A$, and $b^2 \cdot A$ are pairwise disjoint and elements like b^3a are not in their union. Therefore, we have

$$\begin{aligned} 1 = m(1) &\geq m(\chi_{A\cup(b\cdot A)\cup(b^2\cdot A)}) \\ &= m(\chi_A) + m(\chi_{b\cdot A}) + m(\chi_{b^2\cdot A}) \\ &= m(\chi A) + m(b \cdot \chi_A) + m(b^2 \cdot \chi_A) \\ &= 3m(\chi A) \geq 3/2, \end{aligned}$$

which is the desired contradiction.

□

3. More (non)amenable groups

We establish the following theorem to check the amenability of more groups. It turns out that amenability can be inherited in the sense that algebraic operations like taking subgroups, homomorphic images, imposing exact sequences, and decomposing into ascending chains.

Theorem 3.1 (The Inheritance Theorem of Amenability). *(i) Subgroups of amenable groups are amenable.*

(ii) Homomorphic images of amenable groups are amenable.

(iii) *Consider the following group extension (1 denotes the trivial group, i injective, π surjective, and Im $i = \ker \pi$):*

$$1 \longrightarrow N \xrightarrow{\quad i \quad} G \xrightarrow{\quad \pi \quad} Q \longrightarrow 1.$$

Then, G is amenable if and only if N and Q are amenable.

(iv) *For group G, if there is an ascending chain of amenable subgroups, $(G_i)_{i\in I}$, such that $G = \bigcup_{i\in I} G_i$, then G is amenable.*

Before proving the theorem, we make a remark on how to use the theorem and a black-boxed fact that $\mathbb{Z}$ is amenable to show that abelian groups are amenable (Theorem 2.2).

Remark 3.2. By (iii), we note that taking product preserves amenability by viewing products as the extension by the two components. Therefore, using the fundamental theorem of finitely generated abelian groups, we can decompose an arbitrary abelian group G as a product of either $\mathbb{Z}$ or finite cyclic groups. Therefore, by knowing that $\mathbb{Z}$ is amenable, we can deduce Theorem 2.2, which is reassuring since $\mathbb{Z}$ is the building block of all abelian groups.

Proof of Theorem 3.1. **(i)** Let G be amenable with G-invariant mean m. Then, we consider subgroup H and define $m' : \mathcal{L}^\infty(H, \mathbb{R}) \to \mathbb{R}$ as

$$m'(f) = \frac{m(\bar{f})}{\chi_H},$$

where f is a real valued function on H and $\bar{f}$ extends f by sending $G - H$ to 0.

First, notice that when f is the constant 1 function from H to 1, then $\bar{f} = \chi_H$ so that $m'(f_1) = 1$.

Next, when f is non-negative, we note that $\bar{f}$ is also non-negative, which implies $m'(f) \geq 0$.

Finally, let $h \in H$ and we claim that $h \cdot \bar{f} = \overline{h \cdot f}$. Indeed, for $g \in H$, we note that since $hg \in H$, the equality holds since $\bar{f}$ is the extension of f. When $g \notin H$, hg is also not in H so that $h \cdot \bar{f}(g) = \overline{h \cdot f}(g) = 1$. By the claim, we can conclude that

$$m'(h \cdot f) = m(\overline{h \cdot f})/\chi_H = m(h \cdot \bar{f})/\chi_H = m(\bar{f})/\chi_H = m'(f).$$

Thus, H is amenable.

(ii) Suppose that we have a homomorphism $\phi : G \to K$, then $G/\ker \phi \approx \phi(G)$ so that we call the canonical projection

$$\pi : G \to \phi(G).$$

Again, we notice the commutativity between left translation of $\phi(G)$ and π (literally, this is the "left half" of the definition of a group homomorphism), i.e., for $\bar{k} \in \phi(G)$, $\bar{k}\pi(g) = \pi(kg)$.

We define $m' : \mathcal{L}^\infty(\phi(G), \mathbb{R}) \to \mathbb{R}$ again by pre-composing π:

$$m'(f) = m(f \circ \pi).$$

Therefore, we may duplicate the proof in (i) to conclude that m' is a left $\phi(G)$-invariant mean on $\phi(G)$ so that $\phi(G)$ is amenable.

(iii) Note that by the definition of short exact sequences, we may translate the condition to be $N \subset G$ and $Q \approx G/N$. Then, the only if direction directly follows from (i) and (ii).

Conversely, we assume that N has a left invariant mean m_N and Q has a left invariant mean of m_Q. We claim that the following definition of m gives a left invariant mean of G:

$$m(f) = m_Q(h_f),$$

where $h_f \in \mathcal{L}^\infty(G/N, \mathbb{R})$ is defined as

$$h_f(gN) = m_N(k_g : n \mapsto f(gn)),$$

for $f \in \mathcal{L}^\infty(G, \mathbb{R})$, g takes value in a representative set of G/N, and $k_g \in \mathcal{L}^\infty(N, \mathbb{R})$ (we ignore the subscript f on k as an abuse of notation).

You might be intimidated by my definition of m since it involves defining many layers of functions. I hope the definition will become clear in the process of showing that m is a left invariant mean.

First, if f is the constant 1 function, then for any $g \in G$, each k_g is constant one function on N, which would imply that $m_N(k_g) = 1$ for each $g \in G$. Moreover, from the nested definition, we also see that h_f is a constant one function on G/N, which would imply that $m(f) = m_Q(h_f) = 1$.

Second, if f is a non-negative function, we have that each k_g is a non-negative function on N so that $m_N(k) \geq 0$ since m_N is a left-invariant mean. Again, from the nested definition, h_f is a non-negative function so that $m(f) = m_Q(h_f) \geq 0$ since m_Q is a left-invariant mean.

Lastly, assume that $x \in G$ and note that $m(x \cdot f) = m_Q(h_{x \cdot f})$, and the corresponding k_g is defined as $n \mapsto f(xgn)$. Thus, k_g corresponded to $x \cdot f$ is exactly $k_{xg} = x \cdot k_g$ corresponded to f.

Then, we consider two cases.

Case I. $x \in N$. Then, since m_N is left invariant, $m_N(k_g) = m_N(x \cdot k_g)$, which implies that $h_{x \cdot f} = h_f$ so that $m(f) = m(x \cdot f)$.

Case II. $x \notin N$. Observe that $h_{x \cdot f}(gN) = h_f(xgN)$, which means exactly $h_{x \cdot f} = x \cdot h_f$. Since m_Q is left G-invariant, we conclude that

$$m(x \cdot f) = m_Q(h_{x \cdot f}) = m_Q(x \cdot h_f) = m_Q(h_f) = m(f).$$

(iv) We omit this proof, since again, it uses results in functional analysis.

□

To see the power of the inheritance theorem of amenability, we present the following corollaries that tell us amenability of more groups.

Corollary 3.3 (Amenability of locally amenable groups)**.** *A group is amenable if and only if all of its finitely generated subgroups are amenable.*

Proof. The only if direction follows from (i) of Theorem 3.1.

Conversely, if all finitely generated subgroups of G are amenable, then there exists an ascending chain of subgroups of G that cover the entire G (include one more group element at a time). By (iv) of Theorem 3.1, we complete the proof. □

Example 3.4. The infinite dihedral group D_∞ is amenable, since its finitely generated subgroup is either finite (thus amenable) or $\mathbb{Z}$, which is also amenable since it is abelian.

Corollary 3.5 (Solvable groups are amenable)**.** *If a group is solvable, then it is also amenable.*

Proof. First, we know that finite groups are amenable. Recall that if a group G is solvable, then its derived series terminates. Equivalently, G can constructed as an extension of abelian groups (covered in Galois Theory by Mark). Using (iii) of Theorem 3.1, we conclude the proof. □

Example 3.6. Again, from the Galois Theory by Mark, all S_n with $n \leq 4$ are solvable (in fact we checked this in homework). Therefore, all S_n with $n \leq 4$ are amenable. However, we cannot say that about S_5 because of the obvious reason.

Corollary 3.7. *Groups that contain F_2 as a subgroup are not amenable.*

Proof. This directly follows from the contrapositive of (i) of Theorem 3.1 and the fact that F_2 is not amenable. □

Example 3.8. All free groups are not amenable as expected.

Our finial statement is on the amenability of hyperbolic groups. Indeed, we need a fact about hyperbolic group without proof.

Theorem 3.9. *(Ubiquity of free groups in hyperbolic groups) Let G be a hyperbolic group. Then either G is virtually cyclic or G contains a free group of rank 2.*

Returning back to our discussion of amenable groups, we conclude the paper with the following amazing statement on amenability of hyperbolic groups.

Corollary 3.10. *Let G be a hyperbolic group. Then either G is virtually cyclic or G is not amenable.*

Proof. Follows directly from the Ubiquity of free groups in hyperbolic groups. □

4. Bibliography

References

[1] Clara Löh. Geometric Group Theory. Springer International Publishing. 2017.